Nonlocal Kinetic Equations derived from Stochastic Dynamics of Complex Systems

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Stochastic dynamics in continuum

**Definition**

Systems in continuum — finite or countable sets in continuum ($\mathbb{R}^d$)

*cf.* Discrete systems — lattices, graphs

**Interpretation** — particles in mathematical physics, individuals in population ecology, biology, agents on the market in economics

Stochastic dynamics — particles randomly may:

- born (appear)
- die (disappear)
- move (continuously or jump)
Phase space of a system

Definition
The configuration space over $\mathbb{R}^d$ consists of all locally finite subsets from $\mathbb{R}^d$:

$$\Gamma = \Gamma_{\mathbb{R}^d} := \{ \gamma \subset \mathbb{R}^d \mid |\gamma \cap \Lambda| < \infty, \text{ for all } \Lambda \in \mathcal{B}_b(\mathbb{R}^d) \}.$$ 

Here $\mathcal{B}_b(\mathbb{R}^d)$ is the family of all bounded Borel subsets in $\mathbb{R}^d$.

Structures:

- topology
- Borel $\sigma$-algebra $\mathcal{B}(\Gamma)$
- metrization of topology, $\Gamma$ is a Polish space
Statistical description of a system

Let $\mathcal{M}^1_{fm}(\Gamma)$ be the space of probability measures on $(\Gamma, \mathcal{B}(\Gamma))$ with finite local moments:

$$\int_{\Gamma} |\gamma \cap \Lambda|^n d\mu(\gamma) < \infty, \quad \Lambda \in \mathcal{B}_b(\mathbb{R}^d), n \in \mathbb{N}.$$

The description of a system at moment $t \geq 0$ is a distribution (a measure) $\mu_t \in \mathcal{M}^1_{fm}(\Gamma)$.

Example. Chaotic distribution: Poisson measure with intensity $z > 0$

$$\pi_z\left(\{\gamma \in \Gamma \mid |\gamma \cap \Lambda| = n\}\right) = \frac{(zm(\Lambda))^n}{n!} e^{-zm(\Lambda)},$$

$\Lambda \in \mathcal{B}_b(\mathbb{R}^d), n \in \mathbb{N}$. 
Markov evolution

The evolution of a system (the dynamics of measures) is defined via equality

\[
\frac{d}{dt} \int_{\Gamma} F(\gamma) \, d\mu_t(\gamma) = \int_{\Gamma} (LF)(\gamma) \, d\mu_t(\gamma), \quad F : \Gamma \to \mathbb{R},
\]

where \( \mu_0 \in M_{fm}^1(\Gamma) \), \( F \) is from some class of functions \( \mathcal{F}(\Gamma) \).

Operator \( L \) defined on \( \mathcal{F}(\Gamma) \) is (informally) Markovian, i.e.

- \( 1 \in \mathcal{F}(\Gamma) \) and \( L1 = 0 \);
- if \( F \in \mathcal{F}(\Gamma) \) and there exists \( \gamma_0 \in \Gamma \) such that \( F(\gamma) \leq F(\gamma_0) \) for all \( \gamma \in \Gamma \), then \( (LF)(\gamma_0) \leq 0 \).
Examples of Markov generators

- Birth-and-death generator:

\[(LF)(\gamma) := \sum_{x \in \gamma} d(x, \gamma \setminus x) [F(\gamma \setminus x) - F(\gamma)] + \int_{\mathbb{R}^d} b(x, \gamma) [F(\gamma \cup x) - F(\gamma)] dx.\]

- Hopping generator:

\[(LF)(\gamma) := \sum_{x \in \gamma} \int_{\mathbb{R}^d} c(x, y, \gamma \setminus x) [F(\gamma \cup y \setminus x) - F(\gamma)] dy.\]
Birth-and-death spatial processes

A stochastic process behind: $\gamma \mapsto X_t^{\gamma} \overset{d}{\sim} \mu_t$.

History:

- Preston’75 (heuristic);
- Holley/Stroock’78 (finite systems in finite volumes);
- Méléard/... (particular finite systems in $\mathbb{R}^d$);
- Bezborodov’14 (more general finite systems in $\mathbb{R}^d$, PhD Thesis);
- Garsia/Kurtz’06 (infinite systems with $d(x, \gamma) \equiv 1$ and structural restrictions on $b$).
Equilibrium processes

An equilibrium stochastic process: suppose there exists \( \mu \in \mathcal{M}^1_{\text{fm}}(\Gamma) \) such that

\[
\int_{\Gamma} (LF_1)(\gamma)F_2(\gamma) \, d\mu(\gamma) = \int_{\Gamma} F_1(\gamma)(LF_2)(\gamma) \, d\mu(\gamma);
\]

then a.a. \( \gamma^d \sim \mu \) maps to \( X_t^\gamma \sim \mu \).

Kondratiev/Lytvynov'05 (Dirichlet forms approach).

Example: Glauber dynamics

\[
d(x, \gamma) \equiv m > 0, \quad b(x, \gamma) = z \exp\left\{-\sum_{y \in \gamma} \phi(x - y)\right\} > 0.
\]

\( \mu \) is the Gibbs measure \( G_{z, m}, \phi \).

Nonequilibrium dynamics here: the case when \( \mu_0 \neq G_{z, \phi} \).
Correlation functions

Recall that we have

\[ \frac{d}{dt} \int_{\Gamma} F(\gamma) \, d\mu_t(\gamma) = \int_{\Gamma} (LF)(\gamma) \, d\mu_t(\gamma). \]  \hspace{1cm} (1)

Let \( \mu \in M^1_{fm}(\Gamma) \). Suppose that there exists a family of measurable symmetric functions \( k^{(n)}_{\mu} : (\mathbb{R}^d)^n \to [0;+\infty) \), \( n \in \mathbb{N} \), such that

\[ \int_{\Gamma} \sum_{\{x_1,\ldots,x_n\} \subset \gamma} G^{(n)}(x_1,\ldots,x_n) \, d\mu(\gamma) \]

\[ = \frac{1}{n!} \int_{(\mathbb{R}^d)^n} G^{(n)}(x_1,\ldots,x_n) k^{(n)}_{\mu}(x_1,\ldots,x_n) \, dx_1 \ldots dx_n \]

for all symmetric functions \( G^{(n)} \) with bounded supports.
Correlation functions

Then $k^{(n)}_{\mu}$ is called the correlation function (or the factorial moment) of the order $n$ of the measure $\mu$.

Example. For the Poisson measure $\pi_z$, $z > 0$

$$k^{(n)}_{\pi_z}(x_1, \ldots, x_n) \equiv z^n, \quad x_1, \ldots, x_n \in \mathbb{R}^d.$$

Meaning. Let $B_r(y) = \{x \in \mathbb{R}^d \mid |x - y| \leq r\}$. Then

$$k^{(n)}_{\mu}(x_1, \ldots, x_n) = \lim_{r \to 0} \frac{\mu(\{\gamma \in \Gamma \mid |\gamma \cap B_r(x_1)| = \ldots = |\gamma \cap B_r(x_n)| = 1\})}{\left(m(B_r(0))\right)^n}.$$
Evolution of correlation functions

We produce from (1) the evolutional equation for $k_t$:

$$\frac{\partial}{\partial t} k_t = L^\triangle k_t$$

(2)


Based on the so-called harmonic analysis on the configuration spaces: Kondratiev/Kuna’02, Lenard’75.

$$k_t = \begin{pmatrix} k_t^{(0)} \\ k_t^{(1)} \\ \vdots \\ k_t^{(n)} \\ \vdots \end{pmatrix}, \quad L^\triangle = \begin{pmatrix} * & * & \cdots & * & \cdots \\ * & * & \ddots & * & \cdots \\ \vdots & \ddots & \ddots & \ddots & \cdots \\ * & * & \cdots & * & \cdot \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$
**Evolution of correlation functions**

$k_t$ are functions on

$$\Gamma_0 = \left\{ \eta \subset \mathbb{R}^d \mid |\eta| < \infty \right\} \approx \bigsqcup_{n=0}^{\infty} (\mathbb{R}^d)^n$$

$k_t$ should belong to the space

$$\mathcal{K}_C = \left\{ k : \Gamma_0 \rightarrow \mathbb{R} \mid |k(\eta)| \leq \text{const} \cdot C^{|\eta|} \lambda-\text{a.e.} \right\},$$

where $C > 0$ and $\lambda$ is the so-called Lebesgue–Poisson measure, given by

$$\int_{\Gamma_0} H(\eta) \, d\lambda(\eta) = \sum_{n \geq 0} \frac{1}{n!} \int_{(\mathbb{R}^d)^n} H^{(n)}(x_1, \ldots, x_n) \, dx_1 \ldots dx_n.$$
Evolution of correlation functions

A difficulty: $\mathcal{K}_C \approx L^\infty(\Gamma_0, d\lambda)$

$C_0$-semigroup with unbounded generator does not exist there (Lotz’85).

Idea: to consider pre-dual space $L^1(\Gamma_0, C|\eta|d\lambda)$

Kondratiev/Kutoviy/Minlos’08

Realization for weak* solution to (2) for the Glauber dynamics: Kondratiev/Kutoviy/Zhizhina’06.
**Semigroup approach**

Strong solutions to (2) via $C_0$-semigroups: for general birth-and-death generator with a ‘domination of death part’ were obtained in the space

$$\mathcal{K}_{\alpha_C} \subset \mathcal{K}_C, \quad \alpha_0 < \alpha < 1.$$  

F/Kondratiev/Kutoviy’12 J.Funct.Anal.

Technique: analytic semigroups, ⊙-dual semigroups.

Example of ‘death domination’ for the Glauber dynamics:

$$m > \frac{2z}{C} \exp\left\{\int_{\mathbb{R}^d} (1 - e^{-\phi(x)}) \, dx\right\}, \quad \phi \geq 0.$$
Bolker–Pacala dynamics:

\[ d(x, \gamma) = m + \sum_{y \in \gamma} a^-(x - y), \]
\[ b(x, \gamma) = \sum_{y \in \gamma} a^+(x - y), \quad 0 \leq a^\pm \in L^1(\mathbb{R}^d). \]

‘Death domination’:

\[ m > 4C \int_{\mathbb{R}^d} a^-(x) \, dx, \quad Ca^-(x) \geq 4a^+(x), \quad x \in \mathbb{R}^d. \]

Ovsjannikov’s method

Without ‘death domination’ one can work on finite time-interval only: the so-called Ovsjannikov’s method in the scale of spaces

\[ \mathcal{K}_{C'} \subset \mathcal{K}_{C''}, \quad 0 < C_* \leq C' < C'' \leq C^*, \]

if only

\[ \|L^\Delta k\|_{\mathcal{K}_{C''}} \leq \frac{M}{C'' - C'} \|k\|_{\mathcal{K}_{C'}}, \]

then

\[ k_t \in \mathcal{K}_{C_t}, \quad C_t \nearrow \quad t \in [0, T). \]

Glauber dynamics: \( \phi \geq 0 \) and no restrictions on \( m > 0 \)
Aggregation (swarm) model. Another dynamics where Ovsjannikov’s method may be applied:

\[ d(x, \gamma) = m \exp \left\{ - \sum_{y \in \gamma} \phi(x - y) \right\}, \quad b(x, \gamma) \equiv \lambda > 0. \]
The reason of scaling

We know that an evolution of $k_t$ exists. What do we have from this?

We know much more than after an existence theorem about a Markov process on $\Gamma$, namely, we know that

$$k_t^{(n)}(x_1, \ldots, x_n) \leq C_t^n, \quad x_1, \ldots, x_n \in \mathbb{R}^d.$$ 

We know much less than people need in applications: nothing about the ‘exact’ behaviour of $k_t^{(n)}(x_1, \ldots, x_n)$. Even, for $n = 1$, one has

$$\frac{\partial}{\partial t} k_t^{(1)}(x) = \mathcal{L}(k_t^{(1)}, k_t^{(2)}, k_t^{(3)}, \ldots)$$
The idea of a scaling

Idea: to include small parameter $\epsilon$ in the model in such a way that

$$ k_{t,\epsilon}^{(n)}(x_1,\ldots,x_n) = \prod_{i=1}^{n} \rho_t(x_i) + O(\epsilon)(t;x_1,\ldots,x_n), $$

where

$$ \frac{\partial}{\partial t} \rho_t(x) = v(\rho_t)(x). $$

There are different scalings: macroscopic, mesoscopic, diffusive etc.
We consider a mesoscopic scaling: the so-called Vlasov, or mean-field, scaling.

We introduced a modification of the constructions for Hamiltonian dynamics which were proposed by Spohn’80 and, partially, much earlier by Bogolyubov’46.

Namely, we consider $d_\varepsilon(x, \gamma)$ and $\frac{1}{\varepsilon} b_\varepsilon(x, \gamma)$ where all functions kernels, like $a^+, a^-, \phi$ above, are multiplied by $\varepsilon$: $\varepsilon a^\pm, \varepsilon \phi$, and so on. Consider an initial function with the following singularity:

$$\varepsilon^{-|\eta|} r_0(\eta)$$

and apply the dynamics $T_\varepsilon(t) = e^{tL_\varepsilon}$
Vlasov scaling

The idea is to show that the order of singularity will be preserved:

\[ T_\varepsilon(t) \varepsilon^{-|\eta|} r_0(\eta) \sim \varepsilon^{-|\eta|} r_t(\eta), \]

or, equivalently,

\[ T_{\varepsilon,\text{ren}}(t) r_0(\eta) := \varepsilon^{\eta} T_\varepsilon(t) \varepsilon^{-|\eta|} r_0(\eta) \rightarrow r_t(\eta) =: T_{t,V} r_0(\eta), \quad \varepsilon \rightarrow 0, \]

but, the limiting evolution has the significant feature:

\[ r_0(\eta) = \prod_{x \in \eta} \rho_0(x) \quad \text{yields} \quad r_t(\eta) = \prod_{x \in \eta} \rho_t(x), \]

where \( \rho_t \) satisfies the so-called kinetic equation

\[ \frac{\partial}{\partial t} \rho_t(x) = v(\rho_t)(x) \]
**Rigorous constructions**

The point-wise realization of this scheme for general birth-and-death operators (and others) on $\Gamma$ was done in

\[ F/Kondratiev/Kutoviy'10 \text{ J.\textit{Stat. Phys.}} \]

To do this with a full rigour one needs:

- to prove the existence of solutions to
  \[ \frac{d}{dt} k_{t,\varepsilon} = L_{\varepsilon,\text{ren}} k_{t,\varepsilon} \]
  in a functional space (or a scale of spaces), e.g., $\mathcal{K}_C$;

- to prove the existence of solutions to
  \[ \frac{d}{dt} k_{t,V} = L_{V} k_{t,V} \]
  in the same space;

- to prove convergence of solutions, i.e., $T_{\varepsilon,\text{ren}}(t) \to T_V(t)$
  with strong or weak* convergence.
Convergence

For general birth-and-death generators these steps were realized in

F/Kondratiev/Kutoviy’12 J.Funct.Anal.

with weak*-convergence of semigroups.

Strong convergence was obtained, e.g., for the Glauber dynamics on infinite (big \( m > 0 \)) and finite (arbitrary \( m > 0 \)) time-intervals in

F/Kondratiev/Kutoviy’11 IDAQP

F/Kondratiev/Kozitsky’13 DCDS-A

respectively.
Dynamical Kirkwood–Monroe equation

The corresponding kinetic equation is

\[ \frac{\partial}{\partial t}\rho_t(x) = -m\rho_t(x) + ze^{-\phi^*\rho_t}(x), \]

\[ z, m > 0, \quad 0 \leq \phi \in L^1(\mathbb{R}^d), \quad \beta := \int_{\mathbb{R}^d} \phi(x) \, dx. \]

Here and below

\[ (f \ast g)(x) = \int_{\mathbb{R}^d} f(x-y)g(y) \, dy, \quad f \in L^1(\mathbb{R}^d), g \in L^\infty(\mathbb{R}^d). \]

Space-homogeneous stationary solution always exists:

\[ \rho = \frac{z}{m} e^{-\beta \rho}. \]

Stable: for small \( \frac{z}{m}, \beta \iff \left|\mathcal{G}\left(\frac{z}{m}, \phi\right)\right| = 1 \) and ergodicity of microscopic Glauber dynamics.
Dynamical Widom–Rowlinson model

Two-component model: $\gamma^+ \in \Gamma, \gamma^- \in \Gamma$

$$d^+(x, \gamma^+) \equiv m \equiv d^-(x, \gamma^-), \quad b^\pm(x, \gamma^\pm) = z \exp \left\{ - \sum_{y \in \gamma^\pm} \phi(x - y) \right\}.$$ 

The strong convergence $T_{\epsilon,\text{ren}}(t) \to T_V(t)$ in a scale of spaces: 


$$\begin{cases} \frac{\partial}{\partial t} \rho^+_t(x) = -\rho^+_t(x) + z e^{-\phi \rho^-_t}(x) \\ \frac{\partial}{\partial t} \rho^-_t(x) = -\rho^-_t(x) + z e^{-\phi \rho^+_t}(x) \end{cases}$$
Dynamical Widom–Rowlinson model

Consider the space-homogeneous case

\[
\begin{align*}
\frac{d}{dt} \rho^+_t &= -\rho^+_t + ze^{-\beta \rho^-_t} \\
\frac{d}{dt} \rho^-_t &= -\rho^-_t + ze^{-\beta \rho^+_t}
\end{align*}
\]

Theorem

Let \( a := z\beta/m \). If \( a \leq e \) then there exists a unique stationary constant solution \((\frac{1}{\beta} x_0, \frac{1}{\beta} x_0)\), where \( x_0 e^{x_0} = a \). For \( a < e \), this solution is a stable node, while for \( a = e \) it is a saddle-node equilibrium point.
Dynamical Widom–Rowlinson model

Theorem (continuation)

If $a > e$, then there are three and only three stationary constant solutions $\left(\frac{1}{\beta} x_1, \frac{1}{\beta} x_3\right), \left(\frac{1}{\beta} x_2, \frac{1}{\beta} x_2\right), \left(\frac{1}{\beta} x_3, \frac{1}{\beta} x_1\right)$, where $x_1 = a \exp(-x_3), x_2 = a \exp(-x_2), x_3 = a \exp(-x_1)$ and

$$0 < x_1 < a \exp\left(-\frac{a}{e}\right),$$

$$a > x_3 > a \exp\left(-a \exp\left(-\frac{a}{e}\right)\right).$$

The second solution is a saddle point and the other two solutions are stable nodes.
Dynamical Widom–Rowlinson model
Aggregation (swarm) model

Strong convergence $T_{ε,ren}(t) \rightarrow T_V(t)$ in the scale $\{Κ_C\}_{C>0}$.

Kinetic equation:

$$\frac{∂}{∂t}ρ_t(x) = -mρ_t(x) e^{-(ρ_t ϕ)(x)} + \lambda.$$ 

- Existence and uniqueness on finite-time interval.
- For $0 ≤ ρ_0(x) ∈ C_b(\mathbb{R}^d)$, existence on $[0, ∞)$.

First regime

- For $λ < m/(eβ)$, there exist two stationary constant solutions: $κ_1 < κ_2$. Moreover, $κ_1$ is asymptotically and uniformly stable.
- For $λ ≤ m/(eβ)$, $0 ≤ ρ_0 ≤ κ_2$ yields $0 ≤ ρ_t ≤ κ_2$, $t > 0$, and the comparison principle between $κ_1$ and $κ_2$ holds.
Aggregation (swarm) model

Second regime

If $\rho_0$ has a ‘big enough’ pick on an arbitrary domain $A \subset \mathbb{R}^d$:
there exist $b_1, b_2$ (+some conditions) such that

$$b_1 \leq \rho_0(x) \leq b_2, \quad x \in A,$$

then, there exists $a = a(b_1, b_2, A) > 1$, such that

$$b_1 + \frac{\lambda}{a}t \leq \rho_t(x) \leq b_2 + \lambda t, \quad x \in A, t > 0.$$

Moreover, for any $x \in \mathbb{R}^d \setminus A$, there exists time $t(x)$ such that, for $t > t(x)$, $\rho_t(x) \nearrow \infty.$
Second regime

\[ b_2(t) \quad b_1(t) \]

\[ \rho_0 \quad \rho_t \]
Bolker–Pacala model

Ovsjannikov’s method does not work:

\[ \|L^\Delta k\|_{\mathcal{K}_{C''}} \leq \frac{M}{(C'' - C')^2} \|k\|_{\mathcal{K}_{C'}}. \]

Nevertheless, the strong convergence \( T_{\varepsilon,\text{ren}}(t) \to T_V(t) \) in the scale \( \{\mathcal{K}_C\}_{C>0} \) was realized:


Kinetic equation:

\[
\frac{\partial}{\partial t} \rho_t(x) = -m \rho_t(x) + (a^+ \ast \rho_t)(x) - \rho_t(x)(a^- \ast \rho_t)(x)
\]

\[ m > 0, \quad 0 \leq a^\pm \in L^1(\mathbb{R}^d), \quad A^\pm := \int_{\mathbb{R}^d} a^\pm(x) \, dx. \]

For \( m \geq A^+ \): the unique stationary constant solution \( \rho = 0 \). If \( \text{const} \cdot a^+(x) \leq a^-(x) \), then it will be stable.
For $m < A^+$, there exists the second stationary constant solution $\rho_t \equiv \frac{A^+ - m}{A^-} =: \theta$. It will be stable if only

$$a^+(x) \geq \theta a^-(x), \quad x \in \mathbb{R}^d.$$ 

One can rewrite to get a doubly non-local Fisher–KPP equation:

$$\frac{\partial}{\partial t} \rho_t(x) = \int_{\mathbb{R}^d} a^+(x - y)\left(\rho_t(y) - \rho_t(x)\right) dx$$

$$+ \rho_t(x)\left(A^- \theta - (a^- * \rho_t)(x)\right).$$
Doubly non-local Fisher–KPP equation

Non-local Fisher–KPP equations in literature:

\[ \frac{\partial}{\partial t} \rho_t(x) = (\Delta \rho_t)(x) + \rho_t(x)\left(c - (a^- * \rho_t)(x)\right), \]

Fang/Zhao’11, Alfaro/Coville’12, Hamel/Ryzhik’15 etc.

and

\[ \frac{\partial}{\partial t} \rho_t(x) = \int_{\mathbb{R}^d} a^+(x - y)\left(\rho_t(y) - \rho_t(x)\right) dx + \rho_t(x)\left(c - \rho_t(x)\right). \]

Coville/Dávila/Martínez’08, Garnier’11,

Bonnefon/Coville/Garnier/Roques’14 etc.

We prove existence of travelling waves for our equation, find their asymptotic at infinity, and speeds, discover long time behaviour of solutions.
Further examples

Free branching with density dependent establishment:

\[
\frac{\partial}{\partial t}\rho_t(x) = -m\rho_t(x) + (\rho_t \ast a^+)(x)e^{-(\rho_t \ast \phi)(x)}
\]

Free branching with density dependent fecundity:

\[
\frac{\partial}{\partial t}l\rho_t(x) = -m\rho_t(x) + \left((\rho_t e^{-\rho_t \phi}) \ast a^+\right)(x)
\]
Next step

First order approximation:

\[ k_{t,\epsilon}^{(1)}(x) = \rho_t(x) + \epsilon \nu_t(x) + O(\epsilon^2)(t;x). \]
References


