Nonlocal Kinetic Equations *derived* from Stochastic Dynamics of Complex Systems

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Stochastic dynamics in continuum

Definition

Systems in continuum — finite or countable sets in continuum ($\mathbb{R}^d$)

cf. Discrete systems — lattices, graphs

Interpretation — particles in mathematical physics, individuals in population ecology, biology, agents on the market in economics

Stochastic dynamics — particles randomly may:

- born (appear)
- die (disappear)
- move (continuously or jump)

Phase space of a system

Definition

The *configuration space* over $\mathbb{R}^d$ consists of all locally finite subsets from $\mathbb{R}^d$:

$$\Gamma = \Gamma_{\mathbb{R}^d} := \left\{ \gamma \subset \mathbb{R}^d \;\middle|\; |\gamma \cap \Lambda| < \infty, \text{ for all } \Lambda \in \mathcal{B}_{\mathrm{b}}(\mathbb{R}^d) \right\}.$$

Here $\mathcal{B}_{\mathrm{b}}(\mathbb{R}^d)$ is the family of all bounded Borel subsets in $\mathbb{R}^d$.

Structures:

- topology
- Borel σ-algebra $\mathcal{B}(\Gamma)$
- metrization of topology, Γ is a Polish space

Statistical description of a system

Let $\mathcal{M}^1_{\mathrm{fm}}(\Gamma)$ be the space of probability measures on $\big(\Gamma, \mathcal{B}(\Gamma)\big)$ with finite local moments:

$$\int_\Gamma |\gamma \cap \Lambda|^n \, d\mu(\gamma) < \infty, \qquad \Lambda \in \mathcal{B}_{\mathrm{b}}(\mathbb{R}^d), n \in \mathbb{N}.$$

The description of a system at moment $t \geq 0$ is a distribution (a measure) $\mu_t \in \mathcal{M}^1_{\mathrm{fm}}(\Gamma)$.

Example. Chaotic distribution: Poisson measure with intensity $z > 0$

$$\pi_z\Big(\Big\{\gamma \in \Gamma \,\Big|\, |\gamma \cap \Lambda| = n\Big\}\Big) = \frac{\big(zm(\Lambda)\big)^n}{n!} e^{-zm(\Lambda)},$$

$\Lambda \in \mathcal{B}_{\mathrm{b}}(\mathbb{R}^d)$, $n \in \mathbb{N}$.

Markov evolution

The evolution of a system (the dynamics of measures) is defined via equality

$$\frac{d}{dt}\int_\Gamma F(\gamma)\,d\mu_t(\gamma) = \int_\Gamma (LF)(\gamma)\,d\mu_t(\gamma), \quad F:\Gamma\to\mathbb{R},$$

where $\mu_0 \in \mathcal{M}^1_{\mathrm{fm}}(\Gamma)$, F is from some class of functions $\mathcal{F}(\Gamma)$.

Operator L defined on $\mathcal{F}(\Gamma)$ is (informally) Markovian, i.e.

- $1 \in \mathcal{F}(\Gamma)$ and $L1 = 0$;
- if $F \in \mathcal{F}(\Gamma)$ and there exists $\gamma_0 \in \Gamma$ such that $F(\gamma) \le F(\gamma_0)$ for all $\gamma \in \Gamma$, then $(LF)(\gamma_0) \le 0$.

Examples of Markov generators

- Birth-and-death generator:

$$(LF)(\gamma) := \sum_{x\in\gamma} d(x,\gamma\setminus x)\left[F(\gamma\setminus x) - F(\gamma)\right]$$

$$+ \int_{\mathbb{R}^d} b(x,\gamma)\left[F(\gamma\cup x) - F(\gamma)\right] dx.$$

- Hopping generator:

$$(LF)(\gamma) := \sum_{x\in\gamma} \int_{\mathbb{R}^d} c(x,y,\gamma\setminus x)\left[F(\gamma\cup y\setminus x) - F(\gamma)\right] dy.$$

Birth-and-death spatial processes

A stochastic process behind: $\gamma \mapsto X_t^\gamma \overset{d}{\sim} \mu_t$.

History:

- `Preston'75` (heuristic);
- `Holley/Stroock'78` (finite systems in finite volumes);
- `Méléard/...` (particular finite systems in $\mathbb{R}^d$);
- `Bezborodov'14` (more general finite systems in $\mathbb{R}^d$, PhD Thesis);
- `Garsia/Kurtz'06` (infinite systems with $d(x, \gamma) \equiv 1$ and structural restrictions on b).

Equilibrium processes

An equilibrium stochastic process: suppose there exists $\mu \in \mathcal{M}^1_{\mathrm{fm}}(\Gamma)$ such that

$$\int_\Gamma (LF_1)(\gamma)F_2(\gamma)\,d\mu(\gamma) = \int_\Gamma F_1(\gamma)(LF_2)(\gamma)\,d\mu(\gamma);$$

then a.a. $\gamma \overset{d}{\sim} \mu$ maps to $X_t^\gamma \overset{d}{\sim} \mu$

Kondratiev/Lytvynov'05 (Dirichlet forms approach).

Example: Glauber dynamics

$$d(x,\gamma) \equiv m > 0, \quad b(x,\gamma) = z\exp\Big\{-\sum_{y\in\gamma}\phi(x-y)\Big\} > 0.$$

μ is the Gibbs measure $\mathcal{G}_{\frac{z}{m},\phi}$.

Nonequilibrium dynamics here: the case when $\mu_0 \neq \mathcal{G}_{z,\phi}$.

Correlation functions

Recall that we have

$$\frac{d}{dt}\int_\Gamma F(\gamma)\,d\mu_t(\gamma) = \int_\Gamma (LF)(\gamma)\,d\mu_t(\gamma). \qquad (1)$$

Let $\mu \in \mathcal{M}^1_{\mathrm{fm}}(\Gamma)$. Suppose that there exists a family of measurable symmetric functions $k_\mu^{(n)} : (\mathbb{R}^d)^n \to [0;+\infty)$, $n \in \mathbb{N}$, such that

$$\int_\Gamma \sum_{\{x_1,\dots,x_n\}\subset\gamma} G^{(n)}(x_1,\dots,x_n)\,d\mu(\gamma)$$
$$= \frac{1}{n!}\int_{(\mathbb{R}^d)^n} G^{(n)}(x_1,\dots,x_n)k_\mu^{(n)}(x_1,\dots,x_n)\,dx_1\dots dx_n$$

for all symmetric functions $G^{(n)}$ with bounded supports.

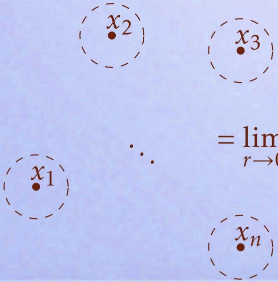
Correlation functions

Then $k_\mu^{(n)}$ is called *the correlation function* (or *the factorial moment*) of the order n of the measure μ.

Example. For the Poisson measure π_z, $z > 0$

$$k_{\pi_z}^{(n)}(x_1, \ldots, x_n) \equiv z^n, \qquad x_1, \ldots, x_n \in \mathbb{R}^d.$$

Meaning. Let $B_r(y) = \{x \in \mathbb{R}^d \mid |x - y| \le r\}$. Then



$$k_\mu^{(n)}(x_1, \ldots, x_n)$$

$$= \lim_{r \to 0} \frac{\mu\Big(\Big\{\gamma \in \Gamma \,\Big|\, |\gamma \cap B_r(x_1)| = \ldots = |\gamma \cap B_r(x_n)| = 1\Big\}\Big)}{\Big(m(B_r(0))\Big)^n}$$

Evolution of correlation functions

We produce from (1) the evolutional equation for k_t:

$$\frac{\partial}{\partial t} k_t = L^{\triangle} k_t \tag{2}$$

F/Kondratiev/Oliveira'09, J.Evol.Eqn.

Based on the so-called harmonic analysis on the configuration spaces: Kondratiev/Kuna'02, Lenard'75.

$$k_t = \begin{pmatrix} k_t^{(0)} \\ k_t^{(1)} \\ \vdots \\ k_t^{(n)} \\ \vdots \end{pmatrix}, \qquad L^{\triangle} = \begin{pmatrix} * & * & \cdots & * & \cdots \\ * & * & \ddots & * & \cdots \\ \vdots & \ddots & \ddots & \ddots & \cdots \\ * & * & \cdots & * & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}$$

Evolution of correlation functions

k_t are functions on

$$\Gamma_0 = \left\{ \eta \subset \mathbb{R}^d \;\middle|\; |\eta| < \infty \right\} \simeq \bigsqcup_{n=0}^{\infty} (\mathbb{R}^d)^n$$

k_t should belong to the space

$$\mathcal{K}_C = \left\{ k : \Gamma_0 \to \mathbb{R} \;\middle|\; |k(\eta)| \le \text{const} \cdot C^{|\eta|} \;\; \lambda\text{-a.e.} \right\},$$

where $C > 0$ and λ is the so-called Lebesgue–Poisson measure, given by

$$\int_{\Gamma_0} H(\eta)\, d\lambda(\eta) = \sum_{n \ge 0} \frac{1}{n!} \int_{(\mathbb{R}^d)^n} H^{(n)}(x_1, \ldots, x_n)\, dx_1 \ldots dx_n.$$

Evolution of correlation functions

A difficulty: $\mathcal{K}_C \approx L^{\infty}(\Gamma_0, d\lambda)$

C_0-semigroup with unbounded generator does not exist there (Lotz′85).

Idea: to consider pre-dual space $L^1(\Gamma_0, C^{|\eta|}d\lambda)$

Kondratiev/Kutoviy/Minlos′08

Realization for weak* solution to (2) for the Glauber dynamics:

Kondratiev/Kutoviy/Zhizhina′06.

Semigroup approach

Strong solutions to (2) via C_0-semigroups: for general birth-and-death generator with a 'domination of death part' were obtained in the space

$$\overline{\mathcal{K}_{\alpha C}}^{\mathcal{K}_C} \subset \mathcal{K}_C, \qquad \alpha_0 < \alpha < 1.$$

F/Kondratiev/Kutoviy'12 J.Funct.Anal.

Technique: analytic semigroups, $\odot$-dual semigroups.

Example of 'death domination' for the Glauber dynamics:

$$m > \frac{2z}{C} \exp\left\{ \int_{\mathbb{R}^d} (1 - e^{-\phi(x)})\, dx \right\}, \quad \phi \geq 0.$$

Bolker–Pacala dynamics

Bolker–Pacala dynamics:

$$d(x,\gamma) = m + \sum_{y\in\gamma} a^-(x-y),$$

$$b(x,\gamma) = \sum_{y\in\gamma} a^+(x-y), \qquad 0 \le a^\pm \in L^1(\mathbb{R}^d).$$

'Death domination':

$$m > 4C \int_{\mathbb{R}^d} a^-(x)\,dx, \qquad C a^-(x) \ge 4a^+(x), \quad x \in \mathbb{R}^d.$$

F/Kondratiev/Kutoviy'09 SIAM J.Math.Anal.

Ovsjannikov's method

Without 'death domination' one can work on finite time-interval only: the so-called Ovsjannikov's method in the scale of spaces

$$\mathcal{K}_{C'} \subset \mathcal{K}_{C''}, \qquad 0 < C_* \leq C' < C'' \leq C^*,$$

if only

$$\left\| L^{\triangle} k \right\|_{\mathcal{K}_{C''}} \leq \frac{M}{C'' - C'} \left\| k \right\|_{\mathcal{K}_{C'}},$$

then

$$k_t \in \mathcal{K}_{C_t} \qquad C_t \nearrow \qquad t \in [0, T).$$

Glauber dynamics: $\phi \geq 0$ and no restrictions on $m > 0$

F/Kondratiev/Kozitsky'13 Discr.Cont.Dyn.Syst.-A

Aggregation model

Aggregation (swarm) model. Another dynamics where Ovsjannikov's method may be applied:

$$d(x,\gamma) = m\exp\Big\{-\sum_{y\in\gamma}\phi(x-y)\Big\}, \qquad b(x,\gamma)\equiv\lambda>0.$$

F/Kondratiev/Kutoviy/Zhizhina'14 Nonlinearity

The reason of scaling

We know that an evolution of k_t exists. What do we have from this?

We know much more than after an existence theorem about a Markov process on Γ, namely, we know that

$$k_t^{(n)}(x_1,\dots,x_n) \le C_t^n, \quad x_1,\dots,x_n \in \mathbb{R}^d.$$

We know much less than people need in applications: nothing about the 'exact' behaviour of $k_t^{(n)}(x_1,\dots,x_n)$. Even, for $n=1$, one has

$$\frac{\partial}{\partial t} k_t^{(1)}(x) = \mathfrak{L}\Big(k_t^{(1)}, k_t^{(2)}, k_t^{(3)}, \dots\Big)$$

The idea of a scaling

Idea: to include small parameter ε in the model in such a way that

$$k_{t,\varepsilon}^{(n)}(x_1,\ldots,x_n) = \prod_{i=1}^{n} \rho_t(x_i) + O(\varepsilon)(t;x_1,\ldots,x_n),$$

where

$$\frac{\partial}{\partial t}\rho_t(x) = v(\rho_t)(x).$$

There are different scalings: macroscopic, mesoscopic, diffusive *etc.*

Vlasov-type scaling

We consider a mesoscopic scaling: the so-called Vlasov, or mean-field, scaling.
We introduced a modification of the constructions for Hamiltonian dynamics which were proposed by Spohn′80 and, partially, much earlier by Bogolyubov′46.
Namely, we consider $d_\varepsilon(x,\gamma)$ and $\frac{1}{\varepsilon} b_\varepsilon(x,\gamma)$ where all functions kernels, like a^+, a^-, ϕ above, are multiplied by ε: $\varepsilon a^\pm$, $\varepsilon\phi$, and so on. Consider an initial function with the following singularity:

$$\varepsilon^{-|\eta|} r_0(\eta)$$

and apply the dynamics $T_\varepsilon(t) = \text{``} e^{tL_\varepsilon^\triangle} \text{''}$

Vlasov scaling

The idea is to show that the order of singularity will be preserved:

$$T_\varepsilon(t)\varepsilon^{-|\eta|}r_0(\eta) \sim \varepsilon^{-|\eta|}r_t(\eta),$$

or, equivalently,

$$T_{\varepsilon,\mathrm{ren}}(t)r_0(\eta) := \varepsilon^{|\eta|}T_\varepsilon(t)\varepsilon^{-|\eta|}r_0(\eta) \to r_t(\eta) =: T_{t,V}r_0(\eta), \quad \varepsilon \to 0,$$

but, the limiting evolution has the significant feature:

$$r_0(\eta) = \prod_{x\in\eta}\rho_0(x) \quad \text{yields} \quad r_t(\eta) = \prod_{x\in\eta}\rho_t(x),$$

where ρ_t satisfies the so-called kinetic equation

$$\frac{\partial}{\partial t}\rho_t(x) = v(\rho_t)(x)$$

Rigorous constructions

The point-wise realization of this scheme for general birth-and-death operators (and others) on Γ was done in

`F/Kondratiev/Kutoviy'10 J.Stat.Phys.`

To do this with a full rigour one needs:

- to prove the existence of solutions to

$$\frac{d}{dt} k_{t,\varepsilon} = L^{\triangle}_{\varepsilon,\mathrm{ren}} k_{t,\varepsilon}$$

  in a functional space (or a scale of spaces), e.g., $\mathcal{K}_C$;

- to prove the existence of solutions to

$$\frac{d}{dt} k_{t,V} = L^{\triangle}_{V} k_{t,V}$$

  in the same space;

- to prove convergence of solutions, i.e., $T_{\varepsilon,\mathrm{ren}}(t) \to T_V(t)$ with strong or weak* convergence.

Convergence

For general birth-and-death generators these steps were realized in

F/Kondratiev/Kutoviy'12 J.Funct.Anal.

with weak*-convergence of semigroups.

Strong convergence was obtained, e.g., for the Glauber dynamics on infinite (big $m > 0$) and finite (arbitrary $m > 0$) time-intervals in

F/Kondratiev/Kutoviy'11 IDAQP

F/Kondratiev/Kozitsky'13 DCDS-A

respectively.

Dynamical Kirkwood–Monroe equation

The corresponding kinetic equation is

$$\frac{\partial}{\partial t}\rho_t(x) = -m\rho_t(x) + ze^{-(\phi*\rho_t)(x)},$$

$$z, m > 0, \quad 0 \le \phi \in L^1(\mathbb{R}^d), \quad \beta := \int_{\mathbb{R}^d} \phi(x)\, dx.$$

Here and below

$$(f*g)(x) = \int_{\mathbb{R}^d} f(x-y)g(y)\, dy, \quad f \in L^1(\mathbb{R}^d), g \in L^\infty(\mathbb{R}^d).$$

Space-homogeneous stationary solution always exists:

$$\rho = \frac{z}{m} e^{-\beta\rho}.$$

Stable: for small $\frac{z}{m}, \beta \iff \left|\left\{\mathcal{G}(\frac{z}{m}, \phi)\right\}\right| = 1$ and ergodicity of microscopic Glauber dynamics.

Dynamical Widom–Rowlinson model

Two-component model: $\gamma^+ \in \Gamma$, $\gamma^- \in \Gamma$

$$d^+(x,\gamma^+) \equiv m \equiv d^-(x,\gamma^-), \qquad b^\pm(x,\gamma^\pm) = z \exp\Big\{-\sum_{y\in\gamma^\mp} \phi(x-y)\Big\}.$$

The strong convergence $T_{\varepsilon,\mathrm{ren}}(t) \to T_V(t)$ in a scale of spaces:

F/Kondratiev/Kutoviy/Oliveira'14 J.Stat.Phys.

$$\begin{cases} \dfrac{\partial}{\partial t}\rho_t^+(x) = -\rho_t^+(x) + ze^{-(\phi*\rho_t^-)(x)} \\ \dfrac{\partial}{\partial t}\rho_t^-(x) = -\rho_t^-(x) + ze^{-(\phi*\rho_t^+)(x)} \end{cases}$$

Dynamical Widom–Rowlinson model

Consider the space-homogeneous case

$$\begin{cases} \dfrac{d}{dt}\rho_t^+ = -\rho_t^+ + ze^{-\beta\rho_t^-} \\ \dfrac{d}{dt}\rho_t^- = -\rho_t^- + ze^{-\beta\rho_t^+} \end{cases}$$

Theorem

Let $a := z\beta/m$. If $a \leq e$ then there exists a unique stationary constant solution $\left(\frac{1}{\beta}x_0, \frac{1}{\beta}x_0\right)$, where $x_0 e^{x_0} = a$. For $a < e$, this solution is a stable node, while for $a = e$ it is a saddle-node equilibrium point.

Dynamical Widom–Rowlinson model

Theorem (continuation)

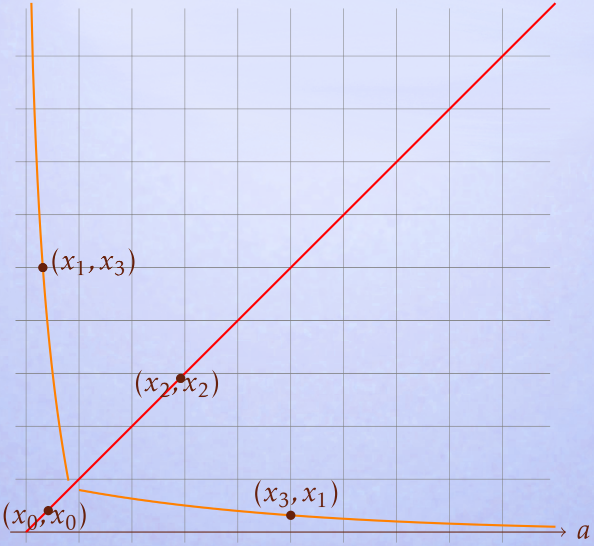
If $a > e$, then there are three and only three stationary constant solutions $\left(\frac{1}{\beta}x_1, \frac{1}{\beta}x_3\right)$, $\left(\frac{1}{\beta}x_2, \frac{1}{\beta}x_2\right)$, $\left(\frac{1}{\beta}x_3, \frac{1}{\beta}x_1\right)$, where $x_1 = a\exp(-x_3)$, $x_2 = a\exp(-x_2)$, $x_3 = a\exp(-x_1)$ and

$$0 < x_1 < a\exp\left(-\frac{a}{e}\right),$$

$$a > x_3 > a\exp\left(-a\exp\left(-\frac{a}{e}\right)\right).$$

The second solution is a saddle point and the other two solutions are stable nodes.

Dynamical Widom–Rowlinson model



Aggregation (swarm) model

Strong convergence $T_{\varepsilon,\mathrm{ren}}(t) \to T_V(t)$ in the scale $\{\mathcal{K}_C\}_{C>0}$.

F/Kondratiev/Kutoviy/Zhizhina'14 Nonlinearity

Kinetic equation:

$$\frac{\partial}{\partial t}\rho_t(x) = -m\rho_t(x)\, e^{-(\rho_t * \phi)(x)} + \lambda.$$

- Existence and uniqueness on finite-time interval.
- For $0 \le \rho_0(x) \in C_b(\mathbb{R}^d)$, existence on $[0,\infty)$.

First regime

- For $\lambda < m/(e\beta)$, there exist two stationary constant solutions: $\kappa_1 < \kappa_2$. Moreover, κ_1 is asymptotically and uniformly stable.
- For $\lambda \le m/(e\beta)$, $0 \le \rho_0 \le \kappa_2$ yields $0 \le \rho_t \le \kappa_2$, $t > 0$, and the comparison principle between κ_1 and κ_2 holds.

Aggregation (swarm) model

Second regime

If ρ_0 has a 'big enough' pick on an arbitrary domain $A \subset \mathbb{R}^d$:
there exist b_1, b_2 (+some conditions) such that

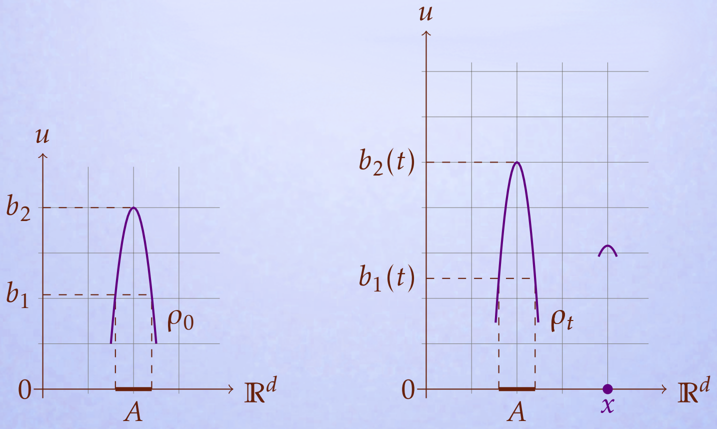
$$b_1 \le \rho_0(x) \le b_2, \quad x \in A,$$

then, there exists $a = a(b_1, b_2, A) > 1$, such that

$$b_1 + \frac{\lambda}{a} t \le \rho_t(x) \le b_2 + \lambda t, \quad x \in A, t > 0.$$

Moreover, for any $x \in \mathbb{R}^d \setminus A$, there exists time $t(x)$ such that,
for $t > t(x)$, $\rho_t(x) \nearrow \infty$.

Second regime



Bolker–Pacala model

Ovsjannikov's method does not work:

$$\left\|L^{\triangle}k\right\|_{\mathcal{K}_{C''}} \le \frac{M}{(C''-C')^2}\left\|k\right\|_{\mathcal{K}_{C'}}.$$

Nevertheless, the strong convergence $T_{\varepsilon,\mathrm{ren}}(t) \to T_V(t)$ in the scale $\{\mathcal{K}_C\}_{C>0}$ was realized:

`F/Kondratiev/Kozitsky/Kutoviy'15 M.Mod.&Meth.Appl.Sci.`

Kinetic equation:

$$\frac{\partial}{\partial t}\rho_t(x) = -m\rho_t(x) + (a^+ * \rho_t)(x) - \rho_t(x)(a^- * \rho_t)(x)$$

$$m > 0, \quad 0 \le a^\pm \in L^1(\mathbb{R}^d), \quad A^\pm := \int_{\mathbb{R}^d} a^\pm(x)\,dx.$$

For $m \ge A^+$: the unique stationary constant solution $\rho = 0$. If $\mathrm{const}\cdot a^+(x) \le a^-(x)$, then it will be stable.

Doubly non-local Fisher–KPP equation

F/Kondratiev/Tkachov In preparation

For $m < A^+$, there exists the second stationary constant solution $\rho_t \equiv \frac{A^+-m}{A^-} =: \theta$. It will be stable if only

$$a^+(x) \geq \theta a^-(x), \quad x \in \mathbb{R}^d.$$

One can rewrite to get a doubly non-local Fisher–KPP equation:

$$\frac{\partial}{\partial t}\rho_t(x) = \int_{\mathbb{R}^d} a^+(x-y)\Big(\rho_t(y) - \rho_t(x)\Big)\,dx$$

$$+\rho_t(x)\Big(A^-\theta - (a^- * \rho_t)(x)\Big).$$

Doubly non-local Fisher–KPP equation

Non-local Fisher–KPP equations in literature:

$$\frac{\partial}{\partial t}\rho_t(x) = (\Delta\rho_t)(x) + \rho_t(x)\Big(c - (a^- * \rho_t)(x)\Big),$$

Fang/Zhao'11, Alfaro/Coville'12, Hamel/Ryzhik'15 *etc.*

and

$$\frac{\partial}{\partial t}\rho_t(x) = \int_{\mathbb{R}^d} a^+(x-y)\Big(\rho_t(y) - \rho_t(x)\Big)\,dx + \rho_t(x)\Big(c - \rho_t(x)\Big).$$

Coville/Dávila/Martínez'08, Garnier'11,

Bonnefon/Coville/Garnier/Roques'14 *etc.*

We prove existence of travelling waves for our equation, find their asymptotic at infinity, and speeds, discover long time behaviour of solutions.

Further examples

Free branching with density dependent establishment:

$$\frac{\partial}{\partial t}\rho_t(x) = -m\rho_t(x) + (\rho_t * a^+)(x)e^{-(\rho_t*\phi)(x)}$$

Free branching with density dependent fecundity:

$$\frac{\partial}{\partial t}l\rho_t(x) = -m\rho_t(x) + \left((\rho_t e^{-\rho_t*\phi}) * a^+\right)(x)$$

Next step

First order approximation:

$$k_{t,\varepsilon}^{(1)}(x) = \rho_t(x) + \varepsilon \nu_t(x) + O(\varepsilon^2)(t;x).$$

Ovaskainen/F/Kutoviy/Cornell/Bolker/Kondratiev'14

Theoretical Ecology

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