Stationary wave on the sphere

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The equation

$$\frac{\partial u}{\partial t} = D\nabla_S^2 u + \Phi(u), \quad x \in [0,\pi],\ t > 0, \tag{1}$$

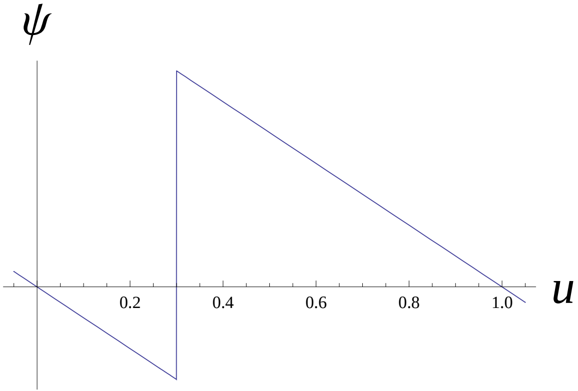
$$\nabla_S^2 := \frac{1}{\sin x}\frac{\partial}{\partial x}(\sin x \frac{\partial}{\partial x}) = \frac{\partial^2}{\partial x^2} + \cot x \frac{\partial}{\partial x}. \tag{2}$$

∇_S^2 - the reduced Laplace-Beltrami operator on the unit sphere in $\mathbb{R}^3$ under the assumption of the axial symmetry.

$$\Psi(u,\sigma) = -Bu + BH(u-\sigma), \tag{3}$$

with $B > 0$, $\sigma \in (0,1)$ and $H : \mathbb{R} \to [0,1]$ the Heaviside function. Below, we will suppose that $D = 1$.

The reaction term



Plot of the function $\Psi(u, \sigma)$ for $\sigma = 0.3$

Plan

- Motivation
- Construction of stationary fronts on the sphere
- Asymptotic behaviour
- Instability properties of the stationary fronts

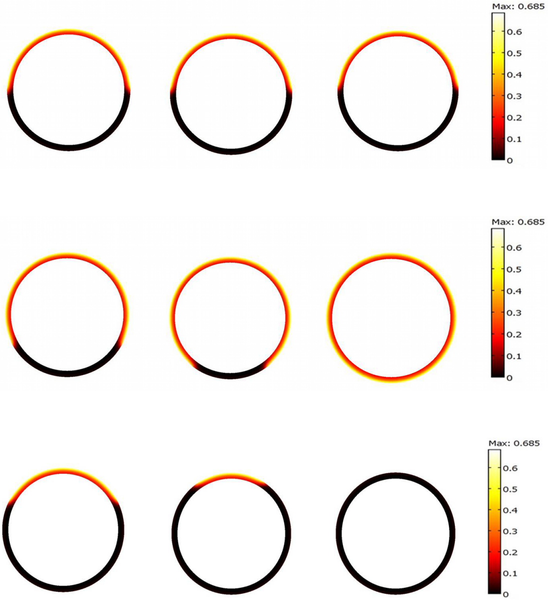
Motivation

B. Hat, B. Kazmierczak, and T. Lipniacki, B cell activation triggered by the formation of the small receptor cluster: A computational study, PLOS Comp. Biol., 5 (2011), e1000448

$$\frac{\partial K(t,x)}{\partial t} = D\nabla_S^2 K \ + aR(1-K) - \frac{bLK}{L+K} \tag{4}$$

$$\frac{\partial R(t,x)}{\partial t} = \gamma[(c_0 + K^2)(P(x) - R) - R] \tag{5}$$

K - surface concentration of the activated membrane kinase molecules, R - concentration of non-diffusing activated membrane receptors.


Max: 0.685
0.6
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Max: 0.685
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Max: 0.685
0.6
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Construction

1. Fix $\eta_0 \in (0, \pi)$. Solve the equations:

$$\frac{\partial}{\partial x}(\sin x \frac{\partial}{\partial x})u + B(1-u) = 0, \quad x \in (0, \eta_0)$$

$$\frac{\partial}{\partial x}(\sin x \frac{\partial}{\partial x})u - Bu = 0, \quad x \in (\eta_0, \pi)$$

with the conditions $\dfrac{\partial u}{\partial x} = 0$ at $x = 0, \pi$.

2. Determine the multiplicative constants to guarantee C^1-smoothness of the obtained solution.

3. Find the correspondence between the parameters η_0 and σ.

Construction. Existence theorem

The function

$$u(x;\eta_0)=\begin{cases}1+C_1(\eta_0)\,F(-\lambda_B,\lambda_B+1,1;\frac{1-\cos x}{2}), & 0<x<\eta_0,\\ C_2(\eta_0)F(-\lambda_B,\lambda_B+1,1;\frac{1+\cos x}{2}), & \eta_0<x<\pi,\\ \sigma, & x=\eta_0,\end{cases}$$

where $\lambda_B := 1/2\left(-1+\sqrt{1-4B}\right)$, satisfies the considered equation. We have:

$$C_1(\eta_0)=C_1(\eta_0;B)=-\frac{1}{c_W}\;F(-\lambda_B+1,\lambda_B+2,2;\frac{1+\cos\,\eta_0}{2})\frac{\sin^2\eta_0}{2},$$
$$C_2(\eta_0)=C_2(\eta_0;B)=\frac{1}{c_W}\;F(-\lambda_B+1,\lambda_B+2,2;\frac{1-\cos\,\eta_0}{2})\frac{\sin^2\eta_0}{2}.$$

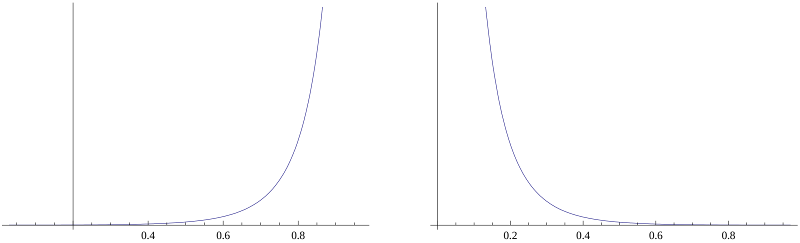
where, for $S=4B-1$,

$$c_W=\frac{2}{\Gamma(-\lambda_B+1)\Gamma(\lambda_B+2)}=8\left(\pi(S+1)\mathrm{sech}\left(\frac{\pi\sqrt{S}}{2}\right)\right)^{-1}.$$

Gauss hypergeometric function

$F(a,b,c,z) := {}_2F_1(a,b,c,z)$ is a hypergeometric function satisfying the hypergeometric equation

$$z(1-z)\frac{d^2w}{dz^2} + [c-(a+b+1)z]\,\frac{dw}{dz} - abw = 0$$

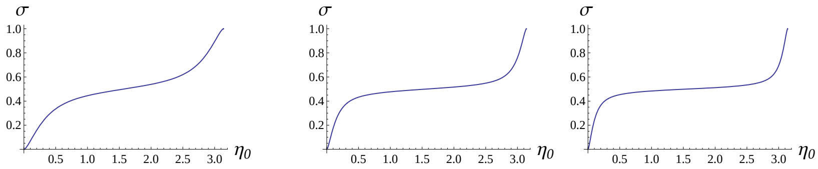


The functions $F(-\lambda_B, \lambda_B+1, 1; z)$ *and* $F(-\lambda_B, \lambda_B+1, 1; 1-z)$ *for* $B = 25$.

Construction

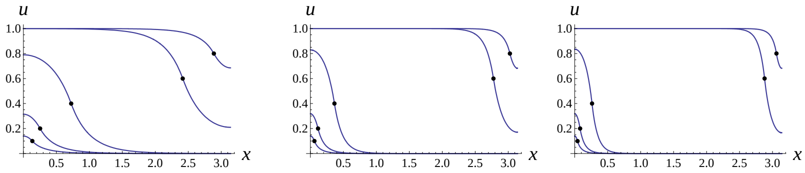
One to one correspondence between η_0 and $\sigma = u(\eta_0; \eta_0)$

The mapping $(0, \pi) \ni \eta_0 \longmapsto u(\eta_0; \eta_0) = \sigma \in (0, 1)$ *is bijective. The value of* $u(\eta_0; \eta_0)$ *is an increasing function of* $\eta_0 \in (0, \pi)$*. More precisely,* $\frac{du(\eta_0;\eta_0)}{d\eta_0} = \frac{d\sigma}{d\eta_0} > 0$ *for all* $\eta_0 \in (0, \pi)$*.*



Relation between between the transition point η_0 *and the parameter* σ *for B: 10, 50 and 100*

Profiles of the stationary waves for different σ and B



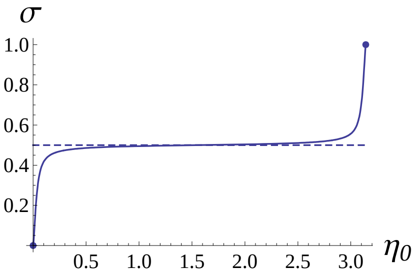
Profiles of the stationary solution for different values of σ: 0.1 (the lowest curves), 0.2, 0.4, 0.6, 0.8 (the highest curves) for different values of B: 10 (left panel), 50 (middle panel) and 100 (right panel). By dots we have denoted the points $(\eta_0(\sigma), \sigma)$.

Asymptotic properties of the stationary waves

Let us fix an arbitrarily small $s_1 > 0$ and set $s_2 = \pi - s_1$. Also let $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ be given. Then the following hold:

i.

For $a, b \in (0, \pi)$ with $a < b$, we have $\sigma(\eta_0; B) \to 1/2$ uniformly for $\eta_0 \in [a, b]$ as $B \to \infty$.

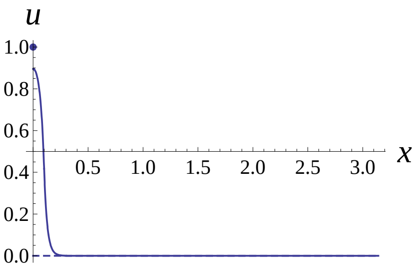


Relation $\sigma(\eta_0)$ for $B = 1000$ (the solid line and the points) and its limit form as $B \to \infty$ (dashed line).

Asymptotic properties of the stationary waves

ii.

For $\sigma \in (0, \sigma_*)$ with $\sigma_* \in (0, 1/2)$, we have $\eta_0(\sigma; B) \in (0, s_1)$ for all B sufficiently large, and $u(x; \eta_0(\sigma; B)) \in (0, \varepsilon_1)$ for all $x \in (\varepsilon_2, \pi)$ and for all B sufficiently large.

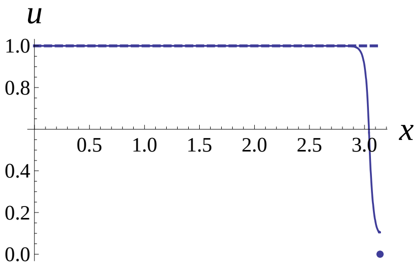


Profiles of the stationary solutions for $B = 1000$ and $\sigma = 0.417$ corresponding to $\eta_0 = 0.1$ (solid line) and its limit as $B \to \infty$ (the dashed line and the point).

Asymptotic properties of the stationary waves

iii.

For $\sigma \in (\sigma^*, 1)$ with $\sigma^* \in (1/2, 1)$, we have $\eta_0(\sigma; B) \in (s_2, \pi)$ for all B sufficiently large, and $u(x; \eta_0(\sigma; B)) \in (1 - \varepsilon_1, 1)$ for all $x \in (0, \pi - \varepsilon_2)$ and for all B sufficiently large.



Profiles of the stationary solutions for $B = 1000$ and $\sigma = 0.583$ corresponding to $\eta_0 = \pi - 0.1$ (solid line) and its limit (the dashed line and the point) as $B \to \infty$.

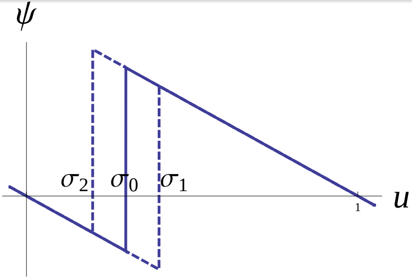
'Separatrix' properties of the stationary fronts

Lemma

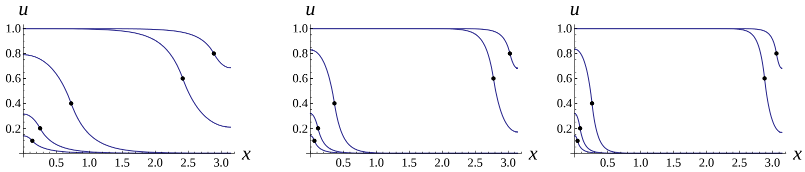
Let $\sigma = \sigma_0 \in (0, 1)$ be fixed. Then for any $\sigma = \sigma_1 > \sigma_0$, $U(x; \sigma_1) > U(x; \sigma_0)$ and is a subsolution to the equation

$$\nabla_S^2 u + \Phi(u; \sigma_0) = 0.$$

Likewise, for any $\sigma = \sigma_2 < \sigma_0$, $U(x; \sigma_1) < U(x; \sigma_0)$ and is a supersolution to this equation.



'Separatrix' properties of the stationary fronts



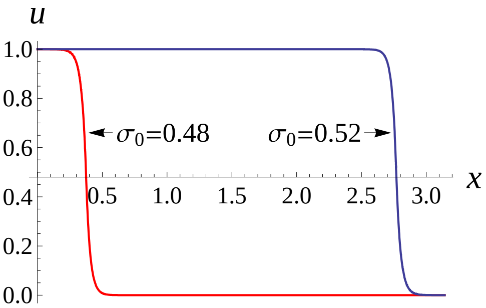
The stationary solutions $U(x;\sigma_0)$ **are not stable.** Given σ_0 and any initial data $u_{in}(x)$ such that $u_{in}(x) > U(x;\sigma_0)$ we can find a subsolution $U_+(x,t;u_{in})$, increasing in t, such that

$$U_+(x,t;u_{in}) \to 1 \quad \text{as} \quad t \to \infty.$$

Likewise, for any initial data $u_{in}(x)$ such that $u_{in}(x) < U(x;\sigma_0)$ we can find a supersolution $U^-(x,t;u_{in})$, decreasing in t, such that

$$U^-(x,t;u_{in}) \to 0 \quad \text{as} \quad t \to \infty.$$

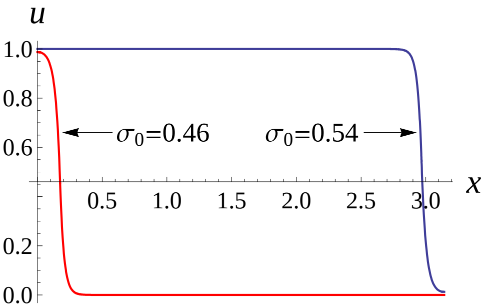
Separatrix of the stationary fronts



The profiles of the stationary solutions $U_0(x;\sigma_0)$ *with* $B = 1000$ *for* $\sigma_0 = 0.52$ *and* $\sigma_0 = 0.48$.

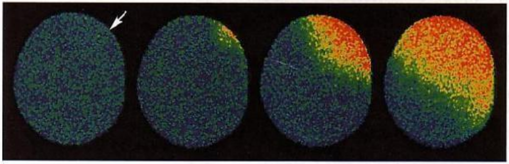
The change in the reaction function may be caused by an external signal, which implies a reorganization inside the cell.

Instability properties of the stationary fronts



The profiles of the stationary solutions $U_0(x;\sigma_0)$ *with* $B = 1000$ *for* $\sigma_0 = 0.54$ *and* $\sigma_0 = 0.46$.

Example: Calcium wave on oocytes



A heteroclinic calcium wave in a starfish egg after fertiliziation. The wave propagates from the animal pole to the vegetal pole of the egg. Red - high Ca concentration.