Evolution of states of a spatial ecological model: Micro- and mesoscopic descriptions

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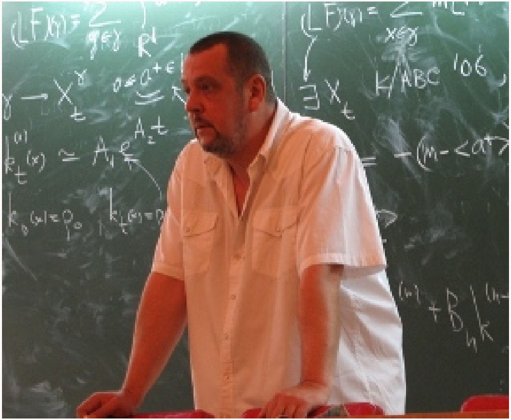
Infinite CPS in life sciences

From physical particles to living entites

http://en.wikipedia.org/wiki/



http://www.math.uni-bielefeld.de/ kondrat/

The basic principles

- *The object:* A large system of interacting entities living in continuous space and evolving in time.
- *The phase space:* A system Γ of infinite sets of identical point particles – configurations, distributed over $\mathbb{R}^d$, $d \geq 1$.
- *The states:* Probability measures on the phase space Γ.
- *The evolution:* Markov-type dynamics of states based on Kolmogorov-Fokker-Planck equations.
- *The elementary events:* Birth, death, jumps, etc.

Evolution

A state is a measure μ on $\Gamma = \{\gamma \subset \mathbb{R}^d : |\gamma \cap \Lambda| < \infty\}$.
Observables: appropriate $F : \Gamma \to \mathbb{R}$

$$\phi(0) \mapsto \phi(t) = \int_\Gamma F_0(\gamma)\mu_t(d\gamma) = \int_\Gamma F_t(\gamma)\mu_0(d\gamma).$$

The Kolmogorov equation

$$\frac{d}{dt}F_t = LF_t, \quad F_t|_{t=0} = F_0,$$

The Fokker-Planck equation

$$\frac{d}{dt}\mu_t = L^*\mu_t, \quad \mu_t|_{t=0} = \mu_0.$$

L and L^* specify the model

Bolker-Pacala spatial ecological (logistic) model

http://www.biology.ufl.edu/people/formermembers.aspx

http://www.iiasa.ac.at/iiasa35/docs/speakers/pacala.html

B. M. Bolker and S. W. Pacala, Using moment equations to understand stochastically driven spatial pattern formation in ecological systems, *Theoret. Population Biol.* **52** (1997) 179–197.

The model

$$\begin{aligned}(LF)(\gamma) &= \sum_{x\in\gamma} D(x,\gamma\setminus x)[F(\gamma\setminus x) - F(\gamma)] \\ &+ \int_{\mathbb{R}^d} B(x,\gamma)[F(\gamma\cup x) - F(\gamma)]dx\end{aligned}$$

$$D(x,\gamma\setminus x) = m + \sum_{y\in\gamma\setminus x} a^-(x-y), \quad B(x,\gamma) = \sum_{y\in\gamma} a^+(x-y).$$

The kernels are assumed

$$a^\pm \in L^1(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d).$$

Correlation functions

States - probability measures on Γ. The Bogoliubov functional

$$B_\mu(\theta) = \int_\Gamma \prod_{x\in\gamma} (1+\theta(x))\, \mu(d\gamma), \quad \theta : \mathbb{R}^d \to (-1, 0].$$

$$B_{\pi_\varkappa}(\theta) = \exp\left(\varkappa \int_{\mathbb{R}^d} \theta(x) dx \right).$$

Sub-Poissonian state: B_μ can be continued to an exponential type entire function of $\theta \in L^1(\mathbb{R}^d)$.

$$B_\mu(\theta) = \sum_{n=0}^{\infty} \frac{1}{n!} \int_{(\mathbb{R}^d)^n} k_\mu^{(n)}(x_1, \dots, x_n)\theta(x_1)\cdots\theta(x_n) dx_1 \cdots dx_n,$$

Correlation functions

Sub-Poisson:

$$\|k_\mu^{(n)}\|_{L^\infty((\mathbb{R}^d)^n)} \le C\exp(-\alpha n), \qquad n \in \mathbb{N}_0,$$

for $C > 0$ and $\alpha \in \mathbb{R}$. The meaning of $k_\mu^{(n)}$ is that, for a $\Lambda \subset \mathbb{R}^d$,

$$\int_{\Lambda^n} k_\mu^{(n)}(x_1, \ldots, x_n) dx_1 \cdots dx_n$$

is the μ-expected number of n-tuples in Λ.

Mean-field approximation

$$k^{(n)}(x_1, \ldots, x_n) \simeq k^{(1)}(x_1) \cdots k^{(1)}(x_n).$$

For $a_- \equiv 0$ (clustering)

$$\text{const} \cdot n! c_t^n \le k^{(n)}(x_1, \ldots, x_n) \le \text{const} \cdot n! C_t^n.$$

Evolution equation

Write $k(\eta) = k^{(n)}(x_1, \ldots, x_n)$ for $\eta = \{x_1, \ldots, x_n\}$, $n \in \mathbb{N}$.

$$(L^\Delta k)(\eta) = -E(\eta)k(\eta) + \sum_{x\in\eta} E^+(x, \setminus x)k(\eta \setminus x)$$

$$-\int_{\mathbb{R}^d} E^-(y,\eta)k(\eta \cup y)dy + \int_{\mathbb{R}^d} \sum_{x\in\eta} a^+(x-y)k(\eta \setminus x \cup y)dy.$$

$$E^\pm(x,\eta) = \sum_{y\in\eta} a^\pm(x-y), \quad E(\eta) = \sum_{x\in\eta} \left(m + E^-(x, \eta \setminus x)\right).$$

The equation

$$\frac{d}{dt}k_t = L^\Delta k_t, \qquad k_0 = k_{\mu_0}.$$

The tasks and the plan

The tasks

- Obtaining evolution of states $\mu_0 \mapsto \mu_t \in \mathcal{P}_{\mathrm{subPoiss}}(\Gamma)$.
- Passing to a mesoscopic level of description.

The plan

- Obtaining evolution $k_{\mu_0} \mapsto k_t \in \mathcal{K}_\alpha$.
- Proving that $\forall t \; \exists! \mu \in \mathcal{P}(\Gamma)$ such that $k_t = k_{\mu_t}$.
- Showing that

$$k^{(n)}_{\varepsilon,t}(x_1, \ldots, x_n) \to \varrho_t(x_1) \cdots \varrho_t(x_n), \quad \varepsilon \to 0,$$

- deriving the (kinetic) equation for ϱ_t.
- studying the solutions of the kinetic equation.

Scale of Banach spaces

Set

$$\|k\|_\alpha = \sup_{n\in\mathbb{N}_0} e^{\alpha n}\|k^{(n)}\|_{L^\infty((\mathbb{R}^d)^n)}, \qquad \alpha \in \mathbb{R},$$

and then

$$\mathcal{K}_\alpha = \{k : \Gamma_0 \to \mathbb{R} : \|k\|_\alpha < \infty\}.$$

Note that

$$\mathcal{K}_\alpha \hookrightarrow \mathcal{K}_{\alpha'} \quad \text{for} \quad \alpha' < \alpha.$$

Set

$$\mathcal{D}_\alpha(L^\Delta) = \{k \in \mathcal{K}_\alpha : L^\Delta k \in \mathcal{K}_\alpha\}.$$

Note that

$$\forall \alpha' > \alpha \qquad \mathcal{K}_{\alpha'} \subset \mathcal{D}_\alpha(L^\Delta).$$

The solution

Definition 1

Given $\alpha \in \mathbb{R}$ *and* $T \in (0, +\infty]$, *by classical solution of*

$$\frac{d}{dt} k_t = L^{\Delta} k_t, \qquad k_0 = k_{\mu_0} \in \mathcal{D}_\alpha(L^{\Delta}), \tag{1}$$

we mean a map $[0, T) \in \mathcal{D}_\alpha(L^{\Delta})$, *which is continuously differentiable on* $[0, T)$ *and solves the equation.*

The basic assumption (BA)

$$\exists \vartheta > 0 \qquad E(\eta) \geq \vartheta E^{+}(\eta), \quad \eta \in \Gamma_0.$$

$$m|\eta| + \sum_{x \in \eta} \sum_{y \in \eta \setminus x} a^{-}(x - y) \geq \vartheta \sum_{x \in \eta} \sum_{y \in \eta \setminus x} a^{+}(x - y).$$

The statement

For a given $\alpha \in \mathbb{R}$, set $\langle a^{\pm} \rangle = \|a^{\pm}\|_{L^1(\mathbb{R}^d)}$ and

$$\mathcal{P}_\alpha = \{\mu \in \mathcal{P}(\Gamma) : k_\mu \in \mathcal{K}_\alpha\}.$$

Theorem 1

Assume BA holds and $\langle a^+ \rangle > 0$. Then for each $\alpha^0 \in \mathbb{R}$, $\mu_0 \in \mathcal{P}_{\mu^0}$, and $T > 0$, the problem

$$\frac{d}{dt} k_t = L^\Delta k_t, \qquad k_0 = k_{\mu_0} \in \mathcal{K}_{\alpha_0},$$

has a unique solution $k_t \in \mathcal{K}^+_{\alpha_T}$ on $[0, T]$, where

$$\alpha_T = \min\{\log \vartheta; \alpha^0\} - T\langle a^+ \rangle. \tag{2}$$

For each such t, there exists a unique $\mu_t \in \mathcal{P}_{\mu_T}$ such that $k_t = k_{\mu_t}$.

More results

If $\langle a^+ \rangle = 0$, then the mentioned solution k_t satisfies the estimate

$$k_t(\eta) \leq \exp\left[-E(\eta)t\right] k_0(\eta), \qquad t > 0, \quad \eta \in \Gamma_0. \tag{3}$$

If $m = 0$ and $a^-(x) = \vartheta a^+(x)$, then the solution

$$k_t(\eta) = \vartheta^{-|\eta|}, \qquad t \geq 0, \tag{4}$$

is a stationary one. Thus, the Poisson measure $\pi_\varkappa$, $\varkappa = 1/\vartheta$ is stationary.

Scaling

Finkelshtein/Kondratiev/K/Kutovyi *Math. Models Methods Appl. Sci.* **25**, 343–370 (2015)

The rescaled equation

$$\frac{d}{dt}k_{\varepsilon,t} = (V + \varepsilon B)k_{\varepsilon,t}, \qquad k_{\varepsilon,0} = r_0 \in \mathcal{K}_{\alpha^0}.$$

The Vlasov equation

$$\frac{d}{dt}r_t = Vr_t, \qquad r_0 = \prod_{x\in\eta} \varrho_0(x).$$

- Both have unique solutions $k_{\varepsilon,t}, r_t \in \mathcal{K}_\alpha \supset \mathcal{K}_{\alpha^0}$ on $[0, T_\alpha)$.
- $\lim_{\varepsilon\to 0} \sup_{s\in[0,t]} \|k_{\varepsilon,t} - r_t\|_\alpha = 0$.
- $r_t(\eta) = \prod_{x\in\eta} \varrho_t(x)$.

$$\frac{d}{dt}\varrho_t = v(\varrho_t), \qquad \varrho_0 \in L^\infty(\mathbb{R}^d).$$

The kinetic equation

$$\frac{d}{dt}\varrho_t(x) = -m\varrho_t(x) - \varrho_t(x)(a^- * \varrho_t)(x) + (a^+ * \varrho_t)(x).$$

Denote

$$\Delta_b^+ = \{\varrho \in L^\infty(\mathbb{R}^d) : \|\varrho\|_{L^\infty(\mathbb{R}^d)} \leq b, \quad \varrho(x) \geq 0\}.$$

$$r(\eta) = \prod_{x\in\eta} \varrho(x), \quad r \in \mathcal{K}_\alpha \;\Leftrightarrow\; \varrho \in \Delta_b^+, \; b = e^{-\alpha}.$$

Theorem 2

For each $\varrho_0 \geq 0$, the kinetic equation has a unique classical solution $\varrho_t \geq 0$ on $[0, +\infty)$ If $\varrho_0 \in C_{\rm b}(\mathbb{R}^d)$ then also $\varrho_t \in C_{\rm b}(\mathbb{R}^d)$ for all $t > 0$.

Properties of the solution

Set

$$\tilde{\Delta}_b^+ = \{\varrho \in \Delta_b^+ : \varrho \in C_{\rm b}(\mathbb{R}^d)\}$$

Theorem 3

Assume $a^+(x) \leq \theta a^-(x)$, $\theta > 0$. Then $\varrho_0 \in \tilde{\Delta}^+_{\theta-\delta}$, $\delta > 0$, implies $\varrho_t \in \tilde{\Delta}^+_{\theta}$. If $m > \langle a^+ \rangle$, then $\varrho_t \to 0$ as $t \to \infty$.

Set

$$q = \frac{\langle a^+ \rangle - m}{\langle a^- \rangle}.$$

Assume $q > 0$ and $\varrho_0(x) = q$. Then $\varrho_t(x) = q$ for all $t > 0$.

Properties of the solution

Theorem 4

Assume there exists $\varkappa^+ > q$ *such that* $a^+(x) \geq \varkappa^+ a^-(x)$ *and*

$$0 < \varkappa^- \leq \varrho_0(x) \leq \varkappa^+, \qquad \varkappa^- \in (0, q).$$

Then

$$\psi_t^- \leq \varrho_t(x) \leq \psi_t^+, \qquad t > 0,$$

and hence $\varrho_t(x) \to q$ *as* $t \to +\infty$.

MANY THANKS