Newton's method for nonlinear stochastic wave equations

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Applications of SWE

- The motion of a strand of DNA[1] [2]

[1] Gonzalez and Maddocks, 2001
[2] Dalang, 2009
[3] Hellemans, 1992
[4] Dalang, 2009

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Introduction

SDE:

$$dX_t = f(t, X_t)dt + g(t, X_t)dW_t, \quad X_0 = \eta.$$

[5] K. Amano, EJDE 2012
[6] M. Wrzosek, EJDE 2012

Introduction

SDE:

$$dX_t = f(t, X_t)dt + g(t, X_t)dW_t, \quad X_0 = \eta.$$

The Newton scheme for SDE[5] [6]:

$$\begin{aligned} dX_t^{(k+1)} &= \left[f(t, X_t^{(k)}) + f_x(t, X_t^{(k)})(X_t^{(k+1)} - X_t^{(k)}) \right] dt \\ &\quad + \left[g(t, X_t^{(k)}) + g_x(t, X_t^{(k)})(X_t^{(k+1)} - X_t^{(k)}) \right] dW_t \\ X_0^{(k+1)} &= \eta. \end{aligned}$$

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Theorem (SDE)

If f and g satisfy the Lipschitz condition in x, then the Newton sequence $(X^{(k)})_{k\in\mathbb{N}}$ converges to the unique solution X:

$$\lim_{k\to\infty} \mathbb{E}\left[\sup_{t\in[0,T]} \left|X_t^{(k)} - X_t\right|^2\right] = 0.$$

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If in addition the Fréchet derivatives f_x and g_x satisfy the Lipschitz condition in x, then

$$P\left(\sup_{t\in[0,T]} \left|X_t^{(k+1)} - X_t^{(k)}\right| \le \rho \Rightarrow \sup_{t\in[0,T]} \left|X_t^{(k+2)} - X_t^{(k+1)}\right| \le R\rho^2\right) \ge 1 - CTR^{-2}$$

for all $R > 0$, $0 < \rho \le 1$, $k = 0, 1, 2, \ldots$

The Newton scheme for IVP first-order SPDE[7]:

$$\frac{\partial}{\partial t}u^{(k+1)}(t,x) + a(t,x)\frac{\partial}{\partial x}u^{(k+1)}(t,x)$$
$$= f\left(t,x,u^{(k)}(t,x)\right) + f_u\left(t,x,u^{(k)}(t,x)\right)\left(u^{(k+1)}(t,x) - u^{(k)}(t,x)\right)$$
$$+ \left[g\left(t,u^{(k)}(t,0)\right) + g_u\left(t,u^{(k)}(t,0)\right)\left(u^{(k+1)}(t,0) - u^{(k)}(t,0)\right)\right]\dot{W}_t$$

$$u^{(k)}(0,x) = \varphi_0(x).$$

[7] M. Wrzosek, CM 2014

Theorem (First-order SPDE)

If f and g satisfy the Lipschitz condition in u, then the Newton sequence $(u^{(k)})_{k\in\mathbb{N}}$ converges to the unique solution u:

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$$+ \left[g\left(t,u^{(k)}(t,0)\right) + g_u\left(t,u^{(k)}(t,0)\right)\left(u^{(k+1)}(t,0) - u^{(k)}(t,0)\right)\right]\dot{W}_t$$

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[8] M. Wrzosek, FDE 2013

Theorem (Parabolic SPDE)

If f and g satisfy the Lipschitz condition in u, then the Newton sequence $(u^{(k)})_{k\in\mathbb{N}}$ converges to the unique solution u:

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for all $R > 0$, $0 < \rho \leq 1$, $k = 0, 1, 2, \ldots$

Time-space white noise

Wiener process[9]

$$W : [0, T] \times \mathbb{R} \to L^2(\Omega)$$

i.e. a Gaussian measure on $\mathcal{B} := \mathcal{B}([0, T] \times \mathbb{R})$

- $W(B) \sim N(0, \lambda(B))$ for $B \in \mathcal{B}$,
- if $B_1 \cap B_2 = \emptyset$ for $B_1, B_2 \in \mathcal{B}$, then
  - $W(B_1)$ and $W(B_2)$ - independent
  - $W(B_1) + W(B_2) = W(B_1 \cup B_2)$.

[9] J. B. Walsh, 1986.

Wave equation

IVP nonlinear stochastic wave equation with nonlocal dependence

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = f\left(t, x, u|_{C_{t,x}}\right) + g(t, x, u|_{C_{t,x}})\dot{W}$$
$$u(0, x) = \phi(x),$$
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where

$$C_{t,x} = \{(s, y) : \ 0 \le s \le t, \ |y - x| \le t - s\}.$$

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where

$$C_{t,x} = \{(s, y) :\ 0 \leq s \leq t,\ |y - x| \leq t - s\}.$$

Denote

$$\|u\|_{t,x}^2 = \mathbb{E}\left[\sup_{(s,y)\in C_{t,x}} |u(s, y)|^2\right] \quad \text{for } t \in [0, T],\ x \in \mathbb{R}.$$

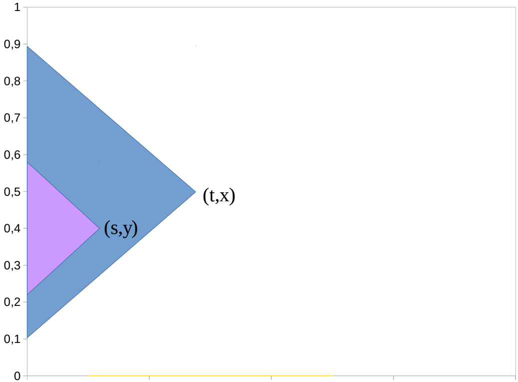

Figure: Wave cones

The d'Alembert formula

$$\begin{aligned} u(t,x) &= \frac{\phi(x-t)+\phi(x+t)}{2} + \frac{1}{2}\int_{x-t}^{x+t} \psi(y)dy \\ &+ \frac{1}{2}\int_{C_{t,x}} f\left(s,y,u|_{C_{s,y}}\right)\, dy\, ds \\ &+ \underbrace{\frac{1}{2}\int_{C_{t,x}} g\left(s,y,u|_{C_{s,y}}\right) W(ds,dy)}_{\text{integral in the sense of Walsh}^{10}\ \text{generalization of Itó}}. \end{aligned}$$

integral in the sense of Walsh[10]
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Gronwall-type inequality

Lemma

Suppose that $\Psi, K : [0, T] \times \mathbb{R} \to \mathbb{R}_+$ *are continuous and increasing w.r.t. cones. If* $z : [0, T] \times \mathbb{R} \to \mathbb{R}_+$ *is continuous and*

$$z(t,x) \leq \frac{1}{2}\int_{C_{t,x}} \Psi(s,y)\,dy\,ds + \frac{1}{2}\int_{C_{t,x}} K(s,y)z(s,y)\,dy\,ds$$

with zero initial conditions

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$$z(t,x) \quad \leq \quad \frac{1}{2} \int_{C_{t,x}} \Psi(s,y)\, dy\, ds + \frac{1}{2} \int_{C_{t,x}} K(s,y) z(s,y)\, dy\, ds$$

with zero initial conditions then

$$z(t,x) \leq \frac{1}{2} e^{t^2 K(t,x)} \int_{C_{t,x}} \Psi(s,y)\, dy\, ds.$$

Estimation of solutions

Lemma

If u satisfies $u(0,x) = \frac{\partial}{\partial t} u(0,x) = 0$ *and*

$$\frac{\partial^2}{\partial t^2} u(t,x) - \frac{\partial^2}{\partial x^2} u(t,x) = \alpha^{(1)}(t,x) + \mathcal{T}^{(1)}(t,x) u|_{C_{t,x}} + \left(\alpha^{(2)}(t,x) + \mathcal{T}^{(2)}(t,x) u|_{C_{t,x}}\right) \dot{W}$$

and $\sup_{|u|\leq 1} \left|\mathcal{T}^{(i)}(t,x)u\right| \leq L(t,x)$

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and $\sup_{|u|\leq 1} \left|\mathcal{T}^{(i)}(t,x)u\right| \leq L(t,x)$ *then*

$$\|u\|_{t,x}^2 \leq K_1(t,x) \int_{C_{t,x}} \left(T^2 \|\alpha^{(1)}\|_{s,y}^2 + N \|\alpha^{(2)}\|_{s,y}^2\right) dy\, ds,$$

$$K_1(t,x) = \frac{1}{2} \exp\left(2t^2 (T^2 + N) L^2(t,x)\right).$$

Sketch of the proof

By d'Alembert's formula

$$\begin{aligned} u(t,x) &= \frac{1}{2}\int_{C_{t,x}} \left(\alpha^{(1)}(s,y) + \mathcal{T}^{(1)}(s,y)u|_{C_{s,y}}\right) dy\, ds \\ &+ \frac{1}{2}\int_{C_{t,x}} \left(\alpha^{(2)}(s,y) + \mathcal{T}^{(2)}(s,y)u|_{C_{s,y}}\right) W(ds,dy). \end{aligned}$$

$$
\begin{aligned}
& |u(t,x)|^2 \\
\leq & \underbrace{\left| \int_{C_{t,x}} \left(\alpha^{(1)}(s,y) + \mathcal{T}^{(1)}(s,y) u|_{C_{s,y}} \right) dy\, ds \right|^2} \\
+ & \underbrace{\left| \int_{C_{t,x}} \left(\alpha^{(2)}(s,y) + \mathcal{T}^{(2)}(s,y) u|_{C_{s,y}} \right) W(ds, dy) \right|^2}
\end{aligned}
$$

$$\sup_{C_{t,x}} \left| u(t,x) \right|^2$$

$$\leq \underbrace{\sup_{C_{t,x}} \left| \int_{C_{t,x}} \left(\alpha^{(1)}(s,y) + \mathcal{T}^{(1)}(s,y) u|_{C_{s,y}} \right) dy\, ds \right|^2}$$

$$+ \underbrace{\sup_{C_{t,x}} \left| \int_{C_{t,x}} \left(\alpha^{(2)}(s,y) + \mathcal{T}^{(2)}(s,y) u|_{C_{s,y}} \right) W(ds,dy) \right|^2}$$

$$\mathbb{E}\left[\sup_{C_{t,x}} |u(t,x)|^2\right]$$

$$\leq \underbrace{\mathbb{E}\left[\sup_{C_{t,x}} \left|\int_{C_{t,x}} \left(\alpha^{(1)}(s,y) + \mathcal{T}^{(1)}(s,y)u|_{C_{s,y}}\right) dy\, ds\right|^2\right]}$$

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The Doob inequality

$$\mathbb{E}\left[\sup_{t\in[0,T]} |\mathcal{M}_t|^p\right] \leq \left(\frac{p}{p-1}\right)^p \mathbb{E}\left[|\mathcal{M}_T|^p\right].$$

[11] Cairoli, 1970
[12] Cairoli, Walsh, 1975
[13] Walsh 2006

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The Cairoli inequality[11] [12] [13]

$$\mathbb{E}\left[\sup_{(s,y)\in C_{t,x}} |\mathcal{M}(s,y)|^p\right] \leq N_p\, \mathbb{E}\left[|\mathcal{M}(t,x)|^p\right].$$

[11] Cairoli, 1970

[12] Cairoli, Walsh, 1975

[13] Walsh 2006

Existence of solutions

Theorem

Under the Lipschitz condition for f and g the sequence of direct iterations $\left(u^{(k)}\right)_{k\in\mathbb{N}}$ converges to the solution u in the following sense

$$\lim_{k\to\infty} \left\| u^{(k)} - u \right\|_{t,x} = 0.$$

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Recall

$$\|u\|_{t,x}^2 = \mathbb{E}\left[\sup_{(s,y)\in C_{t,x}} |u(s,y)|^2 \right].$$

Newton's method

$$\frac{\partial^2}{\partial t^2} u^{(k+1)} - \frac{\partial^2}{\partial x^2} u^{(k+1)}$$
$$= f\left(t, x, u^{(k)}|_{C_{t,x}}\right) + f_u\left(t, x, u^{(k)}|_{C_{t,x}}\right)\left(u^{(k+1)} - u^{(k)}\right)|_{C_{t,x}}$$
$$+ \left[g\left(t, x, u^{(k)}|_{C_{t,x}}\right) + g_u\left(t, x, u^{(k)}|_{C_{t,x}}\right)\left(u^{(k+1)} - u^{(k)}\right)|_{C_{t,x}}\right]\dot{W}$$
$$u^{(k)}(0, x) = \phi(x)$$
$$\frac{\partial}{\partial t} u^{(k)}(0, x) = \psi(x)$$

with

$$u^{(0)}(t, x) = \frac{\phi(x-t) + \phi(x+t)}{2} + \frac{1}{2}\int_{x-t}^{x+t} \psi(y)dy.$$

First-order convergence

Theorem

Suppose that the operators $f_u(t,x)$ and $g_u(t,x)$ are bounded by $L(t,x)$. Then the Newton sequence $\left(u^{(k)}\right)_{k\in\mathbb{N}}$ converges to the unique solution u in the following sense

$$\lim_{k\to\infty} \left\| u^{(k)} - u \right\|_{t,x} = 0.$$

Probabilistic second-order convergence

Theorem

Assume that $\|f_u(t,x)\|^*_{t,x} \le L(t,x), \quad \|g_u(t,x)\|^*_{t,x} \le L(t,x)$

$$\begin{aligned}&\|f_u(t,x,u|_{C_{t,x}}) - f_u(t,x,\bar{u}|_{C_{t,x}})\|^*_{t,x}\\ &+ \|g_u(t,x,u|_{C_{t,x}}) - g_u(t,x,\bar{u}|_{C_{t,x}})\|^*_{t,x}\\ &\qquad \le M(t,x) \sup_{(s,y)\in C_{t,x}} |u(s,y) - \bar{u}(s,y)|\,.\end{aligned}$$

Probabilistic second-order convergence

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Assume that $\|f_u(t,x)\|^*_{t,x} \leq L(t,x), \quad \|g_u(t,x)\|^*_{t,x} \leq L(t,x)$

$$\begin{aligned}&\|f_u(t,x,u|_{C_{t,x}}) - f_u(t,x,\bar{u}|_{C_{t,x}})\|^*_{t,x}\\ &+ \|g_u(t,x,u|_{C_{t,x}}) - g_u(t,x,\bar{u}|_{C_{t,x}})\|^*_{t,x}\\ &\leq M(t,x) \sup_{(s,y)\in C_{t,x}} |u(s,y) - \bar{u}(s,y)|\,.\end{aligned}$$

Then

$$P\left(\sup_{(s,y)\in C_{t,x}} \left|u^{(k+1)}(t,x) - u^{(k)}(t,x)\right| \leq \rho \Rightarrow \sup_{(s,y)\in C_{t,x}} \left|u^{(k+2)}(t,x) - u^{(k+1)}(t,x)\right| \leq R\rho^2\right) \geq 1 - H(t,x)R^{-2}.$$

Sketch of the proof

$$
\begin{aligned}
&\mathbf{1}_{A^k_{\rho,t,x}}\, \epsilon^{(k+1)}(t,x) \\
&= \frac{1}{2}\mathbf{1}_{A^k_{\rho,t,x}} \int_{C_{t,x}} \left(\Lambda_f + f_u \epsilon^{(k+1)}\right)\, dy\, ds \\
&+ \frac{1}{2}\mathbf{1}_{A^k_{\rho,t,x}} \int_{C_{t,x}} \left(\Lambda_g + g_u \epsilon^{(k+1)}\right) W(ds,dy),
\end{aligned}
$$

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$$\mathbf{1}_{A^k_{\rho,t,x}}\, \epsilon^{(k+1)}(t,x)$$
$$= \frac{1}{2}\mathbf{1}_{A^k_{\rho,t,x}} \int_{C_{t,x}} \left(\Lambda_f + f_u \epsilon^{(k+1)}\right) \, dy\, ds$$
$$+ \frac{1}{2}\mathbf{1}_{A^k_{\rho,t,x}} \int_{C_{t,x}} \left(\Lambda_g + g_u \epsilon^{(k+1)}\right) W(ds,dy),$$

where $\epsilon^{(k)} = u^{(k+1)} - u^{(k)}$ and

$$\Lambda_f, \Lambda_g \sim \left(\epsilon^k\right)^2$$
$$A^k_{\rho,t,x} = \left\{\omega : \sup_{(s,y)\in C_{t,x}} |\epsilon^{(k)}(s,y)| \le \rho\right\}.$$

Sketch of the proof

$$\left| \mathbf{1}_{A^k_{\rho,t,x}} \epsilon^{(k+1)}(t,x) \right|^2$$

$$\leq \underbrace{\left| \mathbf{1}_{A^k_{\rho,t,x}} \int_{C_{t,x}} \left(\Lambda_f + f_u \epsilon^{(k+1)} \right) dy\, ds \right|^2}$$

$$+ \underbrace{\left| \mathbf{1}_{A^k_{\rho,t,x}} \int_{C_{t,x}} \left(\Lambda_g + g_u \epsilon^{(k+1)} \right) W(ds,dy) \right|^2}$$

Sketch of the proof

$$\sup_{C_{t,x}} \left| \mathbf{1}_{A^k_{\rho,t,x}} \epsilon^{(k+1)}(t,x) \right|^2$$

$$\leq \underbrace{\sup_{C_{t,x}} \left| \mathbf{1}_{A^k_{\rho,t,x}} \int_{C_{t,x}} \left(\Lambda_f + f_u \epsilon^{(k+1)} \right) dy\, ds \right|^2}$$

$$+ \underbrace{\sup_{C_{t,x}} \left| \mathbf{1}_{A^k_{\rho,t,x}} \int_{C_{t,x}} \left(\Lambda_g + g_u \epsilon^{(k+1)} \right) W(ds, dy) \right|^2}$$

Sketch of the proof

$$\mathbb{E}\left[\sup_{C_{t,x}}\left|\mathbf{1}_{A^k_{\rho,t,x}}\,\epsilon^{(k+1)}(t,x)\right|^2\right]$$

$$\leq \underbrace{\mathbb{E}\left[\sup_{C_{t,x}}\left|\mathbf{1}_{A^k_{\rho,t,x}}\int_{C_{t,x}}\left(\Lambda_f + f_u\epsilon^{(k+1)}\right)dy\,ds\right|^2\right]}$$

$$+ \underbrace{\mathbb{E}\left[\sup_{C_{t,x}}\left|\mathbf{1}_{A^k_{\rho,t,x}}\int_{C_{t,x}}\left(\Lambda_g + g_u\epsilon^{(k+1)}\right)W(ds,dy)\right|^2\right]}$$

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$$+\underbrace{\mathbb{E}\left[\sup_{C_{t,x}}\left|\mathbf{1}_{A^k_{\rho,t,x}}\int_{C_{t,x}}\left(\Lambda_g + g_u\epsilon^{(k+1)}\right)W(ds,dy)\right|^2\right]}$$

Sketch of the proof

$$\mathbb{E}\left[\sup_{C_{t,x}}\left|\mathbf{1}_{A^k_{\rho,t,x}}\,\epsilon^{(k+1)}(t,x)\right|^2\right]$$

$$\leq \underbrace{\mathbb{E}\left[\sup_{C_{t,x}}\left|\mathbf{1}_{A^k_{\rho,t,x}}\int_{C_{t,x}}\left(\Lambda_f + f_u\epsilon^{(k+1)}\right)dy\,ds\right|^2\right]}_{\text{Schwarz inequality OK}}$$

$$+\underbrace{\mathbb{E}\left[\sup_{C_{t,x}}\left|\mathbf{1}_{A^k_{\rho,t,x}}\int_{C_{t,x}}\left(\Lambda_g + g_u\epsilon^{(k+1)}\right)W(ds,dy)\right|^2\right]}_{\text{Cairoli inequality \& isometry WRONG!!!}}$$

Sketch of the proof

$$\left| \mathbf{1}_{A^k_{\rho,t,x}} \epsilon^{(k+1)}(t,x) \right|^2$$

$$\leq \underbrace{\left| \mathbf{1}_{A^k_{\rho,t,x}} \int_{C_{t,x}} \mathbf{1}_{A^k_{\rho,s,y}} \left(\Lambda_f + f_u \epsilon^{(k+1)} \right) dy\, ds \right|^2}$$

$$+ \underbrace{\left| \mathbf{1}_{A^k_{\rho,t,x}} \int_{C_{t,x}} \mathbf{1}_{A^k_{\rho,s,y}} \left(\Lambda_g + g_u \epsilon^{(k+1)} \right) W(ds,dy) \right|^2}$$

Sketch of the proof

$$\sup_{C_{t,x}} \left| \mathbf{1}_{A^k_{\rho,t,x}} \epsilon^{(k+1)}(t,x) \right|^2$$

$$\leq \underbrace{\sup_{C_{t,x}} \left| \mathbf{1}_{A^k_{\rho,t,x}} \int_{C_{t,x}} \mathbf{1}_{A^k_{\rho,s,y}} \left(\Lambda_f + f_u \epsilon^{(k+1)} \right) dy\, ds \right|^2}$$

$$+ \underbrace{\sup_{C_{t,x}} \left| \mathbf{1}_{A^k_{\rho,t,x}} \int_{C_{t,x}} \mathbf{1}_{A^k_{\rho,s,y}} \left(\Lambda_g + g_u \epsilon^{(k+1)} \right) W(ds,dy) \right|^2}$$

Sketch of the proof

$$\mathbb{E}\left[\sup_{C_{t,x}}\left|\mathbf{1}_{A^k_{\rho,t,x}}\,\epsilon^{(k+1)}(t,x)\right|^2\right]$$

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$$+\underbrace{\mathbb{E}\left[\sup_{C_{t,x}}\left|\mathbf{1}_{A^k_{\rho,t,x}}\int_{C_{t,x}}\mathbf{1}_{A^k_{\rho,s,y}}\left(\Lambda_g + g_u\epsilon^{(k+1)}\right)W(ds,dy)\right|^2\right]}$$

Sketch of the proof

$$\mathbb{E}\left[\sup_{C_{t,x}} \left| \mathbf{1}_{A^k_{\rho,t,x}} \epsilon^{(k+1)}(t,x) \right|^2 \right]$$

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$$+ \underbrace{\mathbb{E}\left[\sup_{C_{t,x}} \left| \mathbf{1}_{A^k_{\rho,t,x}} \int_{C_{t,x}} \mathbf{1}_{A^k_{\rho,s,y}} \left(\Lambda_g + g_u \epsilon^{(k+1)}\right) W(ds,dy) \right|^2 \right]}$$

Sketch of the proof

$$\mathbb{E}\left[\sup_{C_{t,x}}\left|\mathbf{1}_{A^k_{\rho,t,x}}\,\epsilon^{(k+1)}(t,x)\right|^2\right]$$

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Example

Consider

$$\begin{aligned}\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} &= \sin\left(u(t/2, x)\right) \dot{W} \\ u(0, x) &= x, \\ \frac{\partial}{\partial t} u(0, x) &= 0.\end{aligned}$$

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For direct iterations we have the estimate

$$\| u^{(k+2)} - u^{(k+1)} \|_{t,x}^2 \leq \frac{\left[(T^2 + N) \right]^{k+1} t^{2(k+1)}}{(k+1)!} N_1 t^2 .$$

Example continued

The corresponding Newton scheme is

$$\frac{\partial^2}{\partial t^2}u^{(k+1)} - \frac{\partial^2}{\partial x^2}u^{(k+1)} = \Big[\sin\left(u^{(k)}(t/2,x)\right)$$
$$+ \cos\left(u^{(k)}(t/2,x)\right)\left(u^{(k+1)}(t/2,x) - u^{(k)}(t/2,x)\right)\Big]\dot{W}$$
$$u^{(k)}(0,x) = x,$$
$$\frac{\partial}{\partial t}u^{(k)}(0,x) = 0.$$

Example continued

We have the estimate

$$\|u^{(k+2)} - u^{(k+1)}\|_{t,x}^2 \leq \frac{\left[2(T^2+N)e^{2t^2(T^2+N)}\right]^{k+1} t^{2(k+1)}}{(k+1)!} N_2 t^2 e^{N_2 t^2}$$

Example continued

We have the estimate

$$\|u^{(k+2)} - u^{(k+1)}\|_{t,x}^2 \leq \frac{\left[2(T^2+N)e^{2t^2(T^2+N)}\right]^{k+1} t^{2(k+1)}}{(k+1)!} N_2 t^2 e^{N_2 t^2}$$

and

$$P\left(\sup_{(s,y)\in C_{t,x}} \left| u^{(k+1)}(t,x) - u^{(k)}(t,x) \right| \leq \rho \Rightarrow \sup_{(s,y)\in C_{t,x}} \left| u^{(k+2)}(t,x) - u^{(k+1)}(t,x) \right| \leq R\rho^2 \right) \geq 1 - T^2(T^2+N)e^{4(T^2+N)t^2} R^{-2}.$$

Thank you for your attention!