Long-time behaviour of the stochastic models of stem cells differentiation.

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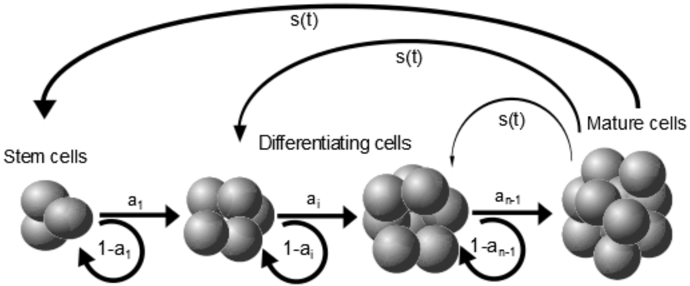

Figure: The process of uni-potential stem cells differentiation

$$
\begin{aligned}
\frac{dc_1}{dt} &= (2a_1 s_{k_1}(t) - 1)p_1 c_1(t) - \mu_1 c_1, \\
\frac{dc_2}{dt} &= (2a_2 s_{k_1}(t) - 1)p_2 c_2(t) + 2(1 - a_1 s_{k_1}(t))p_1 c_1(t) - \mu_2 c_2(t), \\
&\vdots \\
\frac{dc_n}{dt} &= 2(1 - a_{n-1} s_{k_1}(t))p_{n-1} c_{n-1}(t) - \mu_n c_n(t),
\end{aligned} \tag{1}
$$

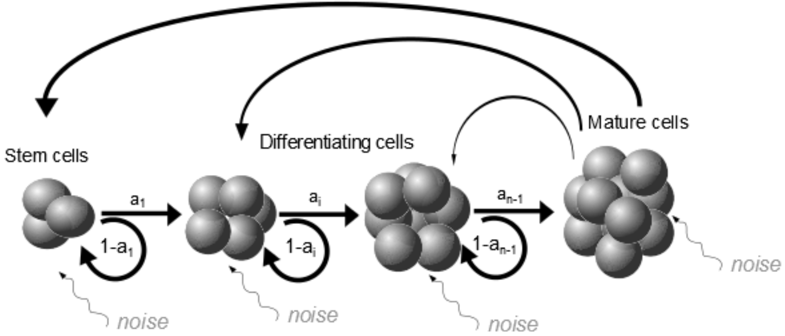

Figure: The process of uni-potential stem cells differentiation with noise

$$
\begin{aligned}
dc_1 &= \left(\left(\frac{2a_1}{1+kc_n} - 1\right) p_1 c_1 - \mu_1 c_1\right) dt \\
dc_2 &= \left(\left(\frac{2a_2}{1+kc_n} - 1\right) p_2 c_2\right. \\
&+ \left. 2\left(1 - \frac{a_1}{1+kc_n}\right) p c_1 - \mu_2 c_2\right) dt \\
&\vdots \\
dc_n &= \left(2\left(1 - \frac{a_{n-1}}{1+kc_n}\right) p_{n-1} c_{n-1} - \mu_n c_n\right) dt
\end{aligned}
\tag{2}
$$

$$
\begin{aligned}
dc_1 &= \left(\left(\frac{2a_1}{1+kc_n} - 1 \right) p_1 c_1 - \mu_1 c_1 \right) dt + \alpha_1 c_1 dW_{1,t} \\
dc_2 &= \left(\left(\frac{2a_2}{1+kc_n} - 1 \right) p_2 c_2 \right. \\
&+ \left. 2 \left(1 - \frac{a_1}{1+kc_n} \right) p c_1 - \mu_2 c_2 \right) dt + \alpha_2 c_2 dW_{2,t} \\
&\vdots \\
dc_n &= \left(2 \left(1 - \frac{a_{n-1}}{1+kc_n} \right) p_{n-1} c_{n-1} - \mu_n c_n \right) dt + \alpha_n c_n dW_{n,t} \qquad (2)
\end{aligned}
$$

$$
\begin{aligned}
dc_1 &= \left(\frac{2a}{1+kc_n} - 1\right) pc_1 dt + \alpha_1 c_1 dW_{1,t} \\
dc_2 &= \left(2\left(1 - \frac{a}{1+kc_2}\right) pc_1 - \mu c_2\right) dt + \alpha_2 c_2 dW_{2,t} \qquad (3)
\end{aligned}
$$

Where $\alpha_1,\ \alpha_2 > 0$.

Theorem

Let the following condition

$$(2a-1)p - \frac{\alpha_1^2}{2} > 0 \tag{4}$$

holds, then the semi-group $\{P(t)\}_{t\geq 0}$ *related to the model* (**??**) *is asymptotically stable.*

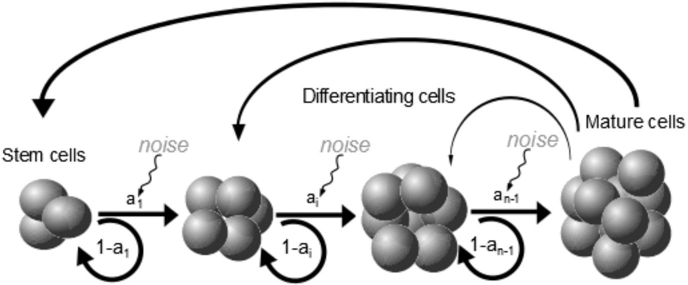

Figure: The process of uni-potential stem cells differentiation with the perturbation of the fraction of self-renewal

$$
\begin{aligned}
x_1'(t) &= \left(\frac{2a}{1+kx_2(t)} - 1\right) p x_1(t) \\
x_2'(t) &= 2\left(1 - \frac{a}{1+kx_2(t)}\right) p x_1(t) - d x_2(t)
\end{aligned} \tag{5}
$$

$$
\begin{aligned}
x_1'(t) &= \left(\frac{2a_t}{1+kx_2(t)} - 1\right) px_1(t) \\
x_2'(t) &= 2\left(1 - \frac{a_t}{1+kx_2(t)}\right) px_1(t) - dx_2(t) \qquad (5)
\end{aligned}
$$

$$
\begin{aligned}
x_1'(t) &= \left(\frac{2a_t}{1+kx_2(t)} - 1\right) p x_1(t) \\
x_2'(t) &= 2\left(1 - \frac{a_t}{1+kx_2(t)}\right) p x_1(t) - d x_2(t)
\end{aligned} \tag{5}
$$

$$
a_t = \{a_0, a_1\}, \quad 0 < a_0 < \frac{1}{2} < a_1 < 1 \tag{6}
$$

$$\begin{aligned} x_1'(t) &= \left(\frac{2a_t}{1+kx_2(t)}-1\right)px_1(t) \\ x_2'(t) &= 2\left(1-\frac{a_t}{1+kx_2(t)}\right)px_1(t)-dx_2(t) \end{aligned} \tag{5}$$

$$a_t=\{a_0,a_1\},\quad 0<a_0<\frac{1}{2}<a_1<1 \tag{6}$$

Theorem

Let $\{P(t)\}_{t\geq 0}$ be the semi-group related to the model (5) and let the following condition

$$\frac{2a_0 - 1}{1 - 2a_1} > \frac{q_0}{q_1} \tag{7}$$

holds. Then the semi-group $\{P(t)\}_{t\geq 0}$ is asymptotically stable.

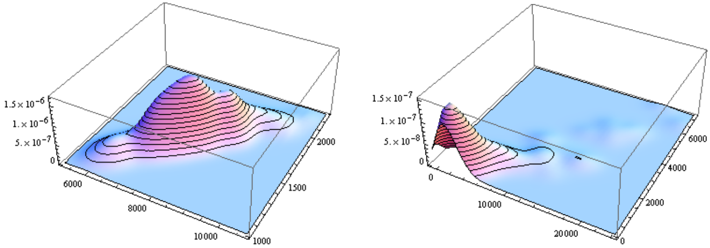

Figure: Densities of the process (3) for $T_{end} = 500$, $p = 0.6$, $\mu = 2.77$, $k = 1,28 \cdot 10^{-6}$ and two sets of α and a: , $a = 0.6$, $\alpha = 0.1$, $a = 0.3$, $\alpha = 1$

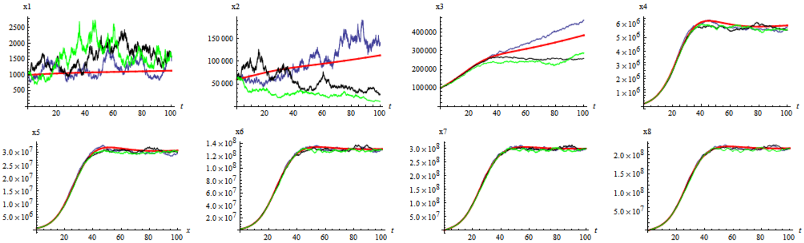

Figure: A solution of the model (2) and 3 trajectories (blue, green, black) of the stochastic processes in time $T_{end} = 100$ of the model (**??**) for $n = 8$ ($\alpha_1 = 1$, $\alpha_2 = 1$, $\alpha_3 = 1$, $\alpha_4 = 0.017$, $\alpha_5 = 0.019$, $\alpha_6 = 0.1$, $\alpha_7 = 0.1$, $\alpha_8 = 0.1$) and $a_1 = 0.735$, $a_2 = 0.7298$, $a_3 = 0.7245$, $a_4 = 0.7140$, $a_5 = 0.5775$, $a_6 = 0.4725$, $a_7 = 0.3675$, $p_1 = 0.006$, $p_2 = 0.03$, $p_3 = 0.18$, $p_4 = 0.6$, $p_5 = 0.65$, $p_6 = 1$, $p_7 = 1.5$, $\mu_1 = \mu_2 = \mu_3 = \mu_4 = \mu_5 = \mu_6 = \mu_7 = 0$, $\mu_8 = 2.77$, $k = 1,28 \cdot 10^{-9}$.

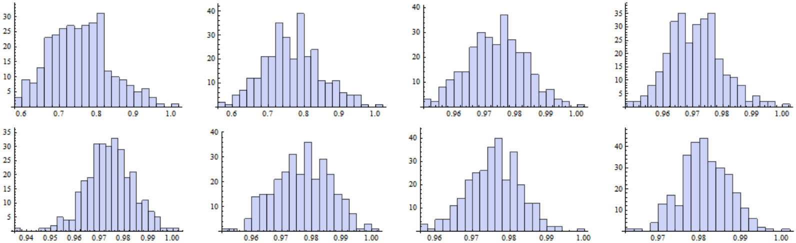

Figure: Histograms for 300 trajectories for a time $T_{end} = 200$ of the model (**??**) for $n = 8$ ($\alpha_1 = 1$, $\alpha_2 = 1$, $\alpha_3 = 1$, $\alpha_4 = 0.017$, $\alpha_5 = 0.019$, $\alpha_6 = 0.1$, $\alpha_7 = 0.1$, $\alpha_8 = 0.1$) and $a_1 = 0.735$, $a_2 = 0.7298$, $a_3 = 0.7245$, $a_4 = 0.7140$, $a_5 = 0.5775$, $a_6 = 0.4725$, $a_7 = 0.3675$, $p_1 = 0.006$, $p_2 = 0.03$, $p_3 = 0.18$, $p_4 = 0.6$, $p_5 = 0.65$, $p_6 = 1$, $p_7 = 1.5$, $\mu_1 = \mu_2 = \mu_3 = \mu_4 = \mu_5 = \mu_6 = \mu_7 = 0$, $\mu_8 = 2.77$, $k = 1{,}28 \cdot 10^{-9}$.

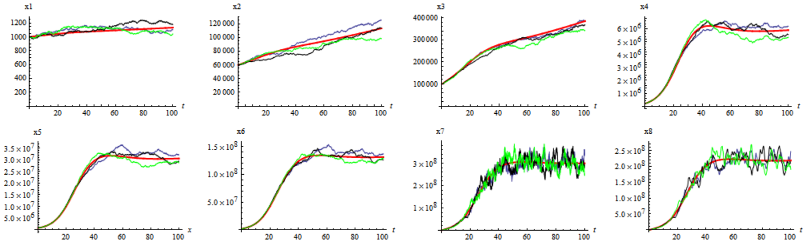

Figure: A solution of the model (2) and 3 trajectories (blue, green, black) of stochastic processes in time $T_{end} = 100$ of the model (**??**) for $n = 8$ with 2 ($\alpha_1 = 0.1$, $\alpha_2 = 0.1$, $\alpha_3 = 0.1$, $\alpha_4 = 0.017$, $\alpha_5 = 0.015$, $\alpha_6 = 0.12$, $\alpha_7 = 1$, $\alpha_8 = 1$) and $a_1 = 0.735$, $a_2 = 0.7298$, $a_3 = 0.7245$, $a_4 = 0.7140$, $a_5 = 0.5775$, $a_6 = 0.4725$, $a_7 = 0.3675$, $p_1 = 0.006$, $p_2 = 0.03$, $p_3 = 0.18$, $p_4 = 0.6$, $p_5 = 0.65$, $p_6 = 1$, $p_7 = 1.5$, $\mu_1 = \mu_2 = \mu_3 = \mu_4 = \mu_5 = \mu_6 = \mu_7 = 0$, $\mu_8 = 2.77$, $k = 1{,}28 \cdot 10^{-9}$.

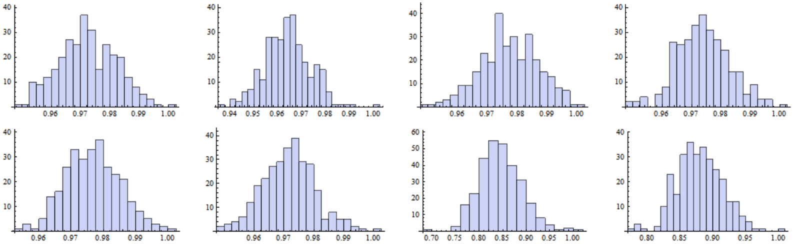

Figure: Histograms for 300 trajectories for a time $T_{end} = 200$ of the model (**??**) for $n = 8$ with 2 ($\alpha_1 = 0.1$, $\alpha_2 = 0.1$, $\alpha_3 = 0.1$, $\alpha_4 = 0.017$, $\alpha_5 = 0.015$, $\alpha_6 = 0.12$, $\alpha_7 = 1$, $\alpha_8 = 1$) and $a_1 = 0.735$, $a_2 = 0.7298$, $a_3 = 0.7245$, $a_4 = 0.7140$, $a_5 = 0.5775$, $a_6 = 0.4725$, $a_7 = 0.3675$, $p_1 = 0.006$, $p_2 = 0.03$, $p_3 = 0.18$, $p_4 = 0.6$, $p_5 = 0.65$, $p_6 = 1$, $p_7 = 1.5$, $\mu_1 = \mu_2 = \mu_3 = \mu_4 = \mu_5 = \mu_6 = \mu_7 = 0$, $\mu_8 = 2.77$, $k = 1{,}28 \cdot 10^{-9}$.

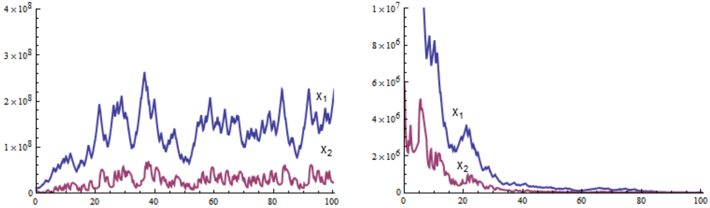

Figure: Trajectories of the processes of the model (5) for $T_{end} = 100$, $p = 0.6$, $\mu = 2.77$, $k = 1,28 \cdot 10^{-6}$, $q_2 = 3$, $q_1 = 1$ and $a_2 = 0.3$, $a_1 = 0.9$ (left one), $a_2 = 0.2$, $a_1 = 0.6$ (right one)

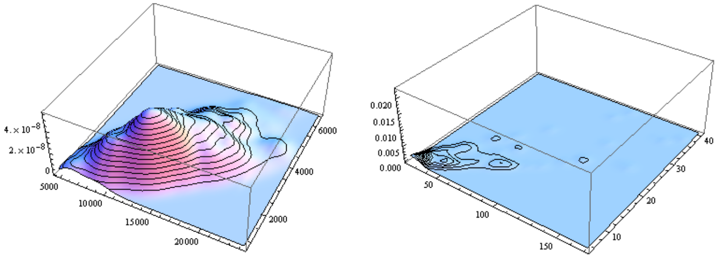

Figure: Density of the process (5) for $T_{end} = 150$ and two different sets of parameters: $a_1 = 0.8$, $a_2 = 0.3$ and $a_1 = 0.6$, $a_2 = 0.2$, $p = 0.6$, $\mu = 2.77$, $k = 1,28 \cdot 10^{-4}$.

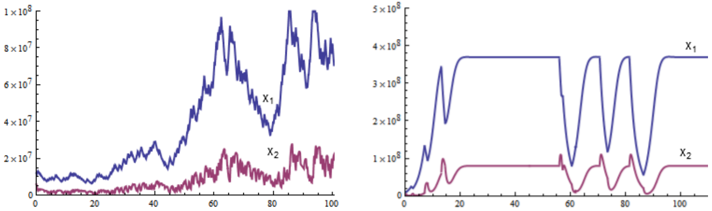

Figure: Trajectories of the processes of the model (5) for $T_{end} = 100$, $p = 0.6$, $\mu = 2.77$, $k = 1,28 \cdot 10^{-6}$, $a_2 = 0.3$, $a_1 = 0.9$ and $q_0 = 4$, $q_1 = 6$ (left one), $q_0 = 0.3$, $q_1 = 0.1$ (right one)

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- Przemys?aw Pa?dziorek *A stochastic perturbation of the fraction of self-renewal in the model of stem cells differentiation*, to appear, see preprint at http://mmns.mimuw.edu.pl/preprints,
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Thank you.