Somatic growth dilution

Toxin predator-prey model under stoichiometric constraints

Angela Peace [1] & Hao Wang [2]

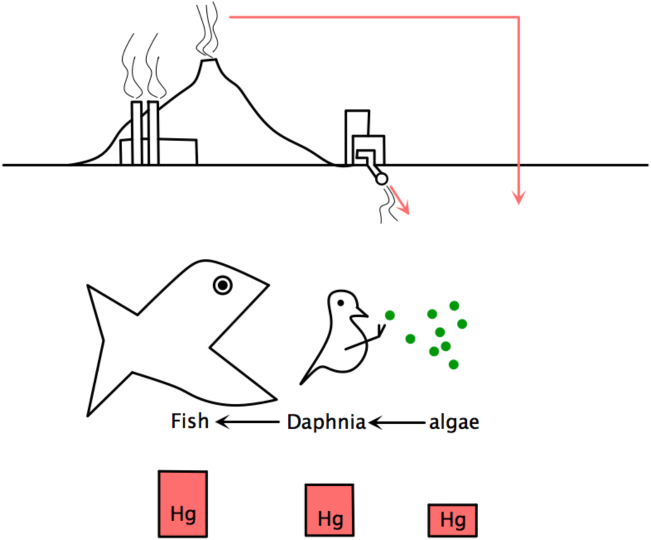
[1]National Institute for Mathematical and Biological Synthesis, University of Tennessee, U.S.A.

[2]Center for Mathematical Biology, Department of Mathematical and Statistical Sciences, University of Alberta, Edmonton, AB, T6G 2G1, Canada

Outline

- Mercury as a toxin and bioaccumulation
- Model formulation: Ecotoxicology and Ecological Stoichiometry
- Model parameterization
- Basic model analysis
- Numerical simulations and bifurcation analysis

Mercury as a toxin



Basic ecotoxicology model

Wang et al. 1996

$$\underbrace{\frac{dv}{dt}}_{\substack{\text{Change in toxin conc.}\\ \text{of predator over time}}} = \underbrace{a_2 T}_{\substack{\text{uptake from}\\ \text{water}}} + \underbrace{\xi f u}_{\substack{\text{uptake from}\\ \text{consuming prey}}} - \underbrace{(\sigma_2 + g)\, v}_{\substack{\text{loss due to}\\ \text{efflux \& growth}}}$$

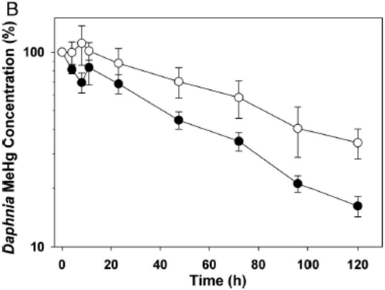
- v predator body burden
- a_2 toxin uptake rate
- T environmental toxin conc.
- u prey body burden
- ξ toxin assimilation efficiency
- f predator's ingestion rate
- σ_2 toxin efflux rate
- g predator's growth rate

Somatic growth dilution

Predator experiences a greater than proportional gain in biomass relative to MeHg under high growth conditions.

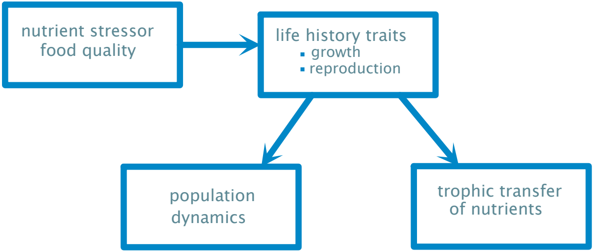
high quality food → rapid growth → lower MeHg concentration

Karimi et al. 2007 showed that *Daphnia* grown on high quality food had 3.5 times higher growth rates and lower MeHg body burden



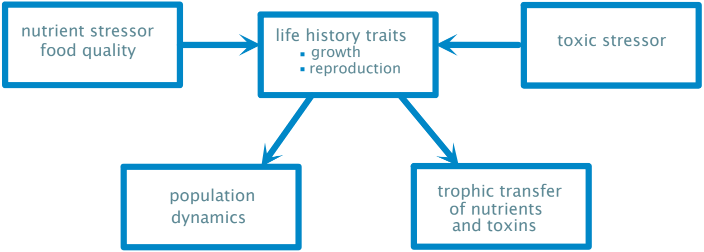
Ecological Stoichiometry and Ecotoxicology

Can Ecological Stoichiometry help improve testing protocols for assessing risk of exposures?



Ecological Stoichiometry and Ecotoxicology

Can Ecological Stoichiometry help improve testing protocols for assessing risk of exposures?



Goal: How does MeHg bioaccumulate under stoichiometric constraints?

Algae*

Daphnia

- Model the trophic transfer of MeHg in aquatic food chain
- Investigate how varying food quality affects toxin bioaccumulation
- Explore dynamics of Somatic Growth Dilution (SGD)

*Image credit: *http : //protist.i.hosei.ac.jp/pdb/images/chlorophyta/scenedesmus*

We start with a Toxin-mediated predator-prey model: Huang et al. 2014

$$\underbrace{\frac{dx}{dt}}_{\text{Change in prey density over time}} = \underbrace{b(u,x)x}_{\text{gain from growth}} - \underbrace{d_1(u)x}_{\text{loss from death}} - \underbrace{f(x)y}_{\text{loss from predation}}$$

$$\underbrace{\frac{dy}{dt}}_{\text{Change in predator density over time}} = \underbrace{e(v)f(x)y}_{\text{gain from growth}} - \underbrace{d_2(v)y}_{\text{loss from death}}$$

$$\underbrace{\frac{du}{dt}}_{\text{Change in prey body burden over time}} = \underbrace{a_1 T}_{\text{uptake from dissolved toxin}} - \underbrace{\sigma_1 u}_{\text{efflux}} - \underbrace{b(u,x)u}_{\text{loss due to growth}}$$

$$\underbrace{\frac{dv}{dt}}_{\text{Change in predator body burden over time}} = \underbrace{a_2 T}_{\text{uptake from dissolved toxin}} - \underbrace{\sigma_2 v}_{\text{efflux}} - \underbrace{e(v)f(x)v}_{\text{loss due to growth}} + \underbrace{f(x)u}_{\text{gain from predation}}$$

We start with a Toxin-mediated predator-prey model: Huang et al. 2014

$$\underbrace{\frac{dx}{dt}}_{\text{Change in prey density over time}} = \underbrace{b(u,x)x}_{\text{gain from growth}} - \underbrace{d_1(u)x}_{\text{loss from death}} - \underbrace{f(x)y}_{\text{loss from predation}}$$

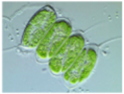
$$\underbrace{\frac{dy}{dt}}_{\text{Change in predator density over time}} = \underbrace{e(v)f(x)y}_{\text{gain from growth}} - \underbrace{d_2(v)y}_{\text{loss from death}}$$

$$\underbrace{\frac{du}{dt}}_{\text{Change in prey body burden over time}} = \underbrace{a_1 T}_{\text{uptake from dissolved toxin}} - \underbrace{\sigma_1 u}_{\text{efflux}} - \underbrace{b(u,x)u}_{\text{loss due to growth}}$$

$$\underbrace{\frac{dv}{dt}}_{\text{Change in predator body burden over time}} = \underbrace{a_2 T}_{\text{uptake from dissolved toxin}} - \underbrace{\sigma_2 v}_{\text{efflux}} - \underbrace{e(v)f(x)v}_{\text{loss due to growth}} + \underbrace{f(x)u}_{\text{gain from predation}}$$

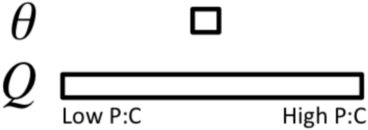
Expand under the Ecological Stoichiometry Framework

Stoichiometric compositions



$Q = P : C$

$\theta = P : C$



- Composition of algae depends on light and nutrient availability
- Varying food quality can influence how MeHg bioaccumulates
- $Q = \frac{P - \theta y}{x}$
- Prey growth $b(u, x)$ and predator conversion efficiency $e(v)$ depend on nutrient availability

$$b(u, x) \rightarrow b(u, x, y) \qquad e(v) \rightarrow e(v, x, y)$$

Prey growth

$$b(u,x,y) = \alpha_1 \max\{0, 1-\alpha_2 u\} \left(1 - \frac{x}{\min\left\{K, \frac{P-\theta y}{q}\right\}}\right)$$

- x prey density
- y predator density
- u prey body burden
- α_1 maximum prey growth rate
- α_2 toxin affect on prey growth
- K producer carrying capacity
- P total phosphorus in the system
- q producer minimal P:C
- θ grazer's constant P:C

Predator conversion efficiency

$$e(v, x, y) = \min\left\{\beta_1, \frac{Q}{\theta}\right\} \max\{0, 1 - \beta_2 v\}$$

- x prey density
- y predator density
- v predator body burden
- β_1 predator C growth efficiency
- β_2 toxin affect on predator growth
- P total phosphorus in the system
- θ grazer's constant P:C
- $Q = \frac{P - \theta y}{x}$ producer varying P:C

The stoichiometric toxin-mediated predator-prey system takes the final form:

$$\underbrace{\frac{dx}{dt}}_{\text{Change in prey}} = \underbrace{\alpha_1 \max\{0, 1-\alpha_2 u\}\left(1 - \frac{x}{\min\left\{K, \frac{P-\theta y}{q}\right\}}\right)x}_{\text{gain from growth}} - \underbrace{f(x)y}_{\text{loss from predation}}$$

$$\underbrace{\frac{dy}{dt}}_{\text{Change in predator}} = \underbrace{\min\left\{\beta_1, \frac{Q}{\theta}\right\}\max\{0, 1-\beta_2 v\}f(x)y}_{\text{gain from growth}} - \underbrace{d_2(v)y}_{\text{loss from death}}$$

$$\underbrace{\frac{du}{dt}}_{\text{Change in toxin conc. of prey}} = \underbrace{a_1 T}_{\text{uptake}} - \underbrace{\sigma_1 u}_{\text{efflux}} \underbrace{- \alpha_1 \max\{0, 1-\alpha_2 u\}\left(1 - \frac{x}{\min\left\{K, \frac{P-\theta y}{q}\right\}}\right)u}_{\text{loss due to growth}}$$

$$\underbrace{\frac{dv}{dt}}_{\text{Change in toxin conc. of predator}} = \underbrace{a_2 T}_{\text{uptake}} + \underbrace{\xi f(x)u}_{\text{uptake from consuming prey}} - \underbrace{\sigma_2 v}_{\text{efflux}} - \underbrace{\min\left\{\beta_1, \frac{Q}{\theta}\right\}\max\{0, 1-\beta_2 v\}f(x)v}_{\text{loss due to growth}}$$

Model parameterization

α_1	Algae maximal growth rate	1.2/day	
α_2	Toxin effect on algal reproduction	0.0051 mg C/μg T	$\star$
K	Algae C carrying capacity	0-3 mg C/L	
β_1	*Daphnia* C growth efficiency	0.8 (unitless)	
β_2	Toxin effect on *Daphnia* reproduction	10.13 mg C/ μg T	$\star$
θ	*Daphnia* constant P:C	0.03 mg P/mg C	
q	Algae minimal P:C	0.0038 mg P/mg C	
h_2	Toxin Coefficient for *Daphnia* mortality	64 mg C/μg T/day	$\star$
l	Toxin Exponent for *Daphnia* mortality	1.17 (unitless)	$\star$
m_2	*Daphnia* natural mortality	0.25/day	
c	*Daphnia* max ingestion rate	0.81/day	
a	*Daphnia* ingestion half saturation constant	0.25 mgC/L	
a_1	Algae uptake coefficient	0.012 L/mg C/day	
a_2	*Daphnia* uptake coefficient	0.011 L/mg C/day	
σ_1	Algae toxin efflux rate	0.048/day	
σ_2	*Daphnia* toxin efflux rate	0.04/day	
ξ	*Daphnia* toxin assimilation efficiency	0.97 (unitless)	
T	Total Toxin	μg MeHg / L	
P	Total phosphorus	mg 0.01-0.08 mg P/ L	

α_2: toxin effect on algal reproduction

$$b(u, x, y) = \alpha_1 \max\{0, 1 - \alpha_2 u\} \left(1 - \frac{x}{\min\left\{K, \frac{P-\theta y}{q}\right\}}\right)$$

To estimate α_2 we use

$$\frac{1}{\alpha_2} = \frac{a_1}{\sigma_1} T_0$$

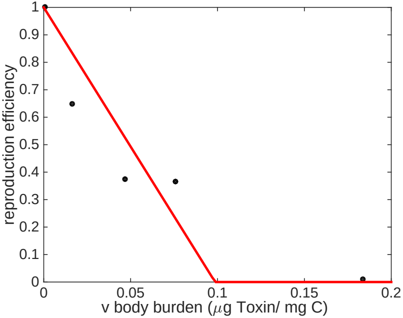
where

- a_1 algal MeHg uptake rate
- σ_1 algal MeHg efflux rate
- T_0 MeHg conc. that inhibits growth 100%
- α_2 MeHg affect on algal growth

β_2: toxin effect on *Daphnia* reproduction

$$e(v, x, y) = \min\left\{\beta_1, \frac{Q}{\theta}\right\} \max\{0, 1 - \beta_2 v\}$$

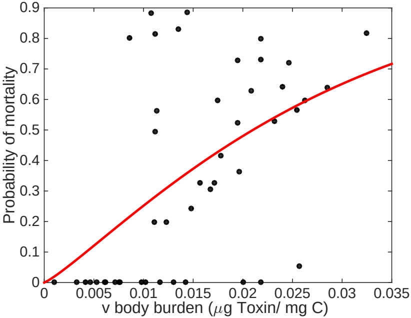
We fit data presented by *Biesinger et al. 1982* on the average number of neonates produced by *Daphnia magna* throughout 21 days of exposure to MeHg



$d_2(v)$: *Daphnia* mortality

$$d_2(v) = h_2 v^l + m_2$$

We fit data presented by *Tsui et al. 2006* on the percent survival of juvenile *Daphnia magna* after 24 hours of exposure to treatments of 1.5-7μg Hg / L.



Nondimensionalization

$$\widetilde{u} = \alpha_2 u, \qquad \widetilde{m_2} = \frac{m_2}{\alpha_1}, \qquad \widetilde{\beta}_1 = \frac{c\beta_1}{\alpha_1}, \qquad \widetilde{\beta}_2 = \frac{\xi c \sigma_1 \beta_2}{\alpha_2}, \qquad \gamma = \frac{a_2\beta_2}{\alpha_2 a_1},$$

$$\widetilde{v} = \beta_2 v, \qquad \epsilon = \alpha_1\sigma_1, \qquad \widetilde{t} = \alpha_1 t, \qquad \widetilde{\sigma_2} = \sigma_2\sigma_1, \qquad \widetilde{T} = \alpha_2 a_1 \sigma_1 T,$$

$$\widetilde{y} = \frac{c}{\alpha_1} y, \qquad \widetilde{h}_1 = \frac{h_1}{\alpha_1\alpha_2}, \qquad \widetilde{h}_2 = \frac{h_2}{\beta_2\alpha_1}, \qquad \widetilde{\theta} = \frac{\alpha_1\theta}{c}, \qquad \widetilde{Q} = \frac{P - \widetilde{\theta}\widetilde{y}}{x}.$$

Dropping the tildes, the system can be written:

$$\frac{dx}{dt} = \max\{0, 1-u\}\left(1 - \frac{x}{\min\left\{K, \frac{P-\theta y}{q}\right\}}\right)x - \frac{xy}{a+x}$$

$$\frac{dy}{dt} = \min\left\{\beta_1, \frac{Q}{\theta}\right\}\max\{0, 1-v\}\frac{xy}{a+x} - (h_2 v + m_2)\, y$$

$$\epsilon\frac{du}{dt} = T - \sigma_2^2 u - \epsilon\max\{0, 1-u\}\left(1 - \frac{x}{\min\left\{K, \frac{P-\theta y}{q}\right\}}\right)u$$

$$\epsilon\frac{dv}{dt} = \gamma T - \sigma_2 v + \left[\beta_2 u - \epsilon\min\left\{\beta_1, \frac{Q}{\theta}\right\}\max\{0, 1-v\}v\right]\frac{x}{a+x}$$

Quasi-steady state assumption: $\epsilon \to 0$

$$u = \frac{T}{\sigma_1^2}, \qquad v = \frac{T}{\sigma_2}\left(\gamma + \frac{\beta_2}{\sigma_1^2}\frac{x}{a+x}\right)$$

The reduced system becomes:

$$\frac{dx}{dt} = \max\left\{0, 1 - \frac{T}{\sigma_1^2}\right\}\left(1 - \frac{x}{\min\left\{K, \frac{P-\theta y}{q}\right\}}\right)x - \frac{xy}{a+x}$$

$$\frac{dy}{dt} = \min\left\{\beta_1, \frac{Q}{\theta}\right\}\max\left\{0, 1 - \frac{T}{\sigma_2}\left(\gamma + \frac{\beta_2}{\sigma_1^2}\frac{x}{a+x}\right)\right\}\frac{xy}{a+x} - \left(\frac{h_2 T}{\sigma_2}\left(\gamma + \frac{\beta_2}{\sigma_1^2}\frac{x}{a+x}\right) + m_2\right)y$$

which can be written as:

$$\frac{dx}{dt} = xF(x, y)$$
$$\frac{dy}{dt} = yG(x, y)$$

Boundedness and positive invariance

Solutions to reduced system with initial conditions in the set

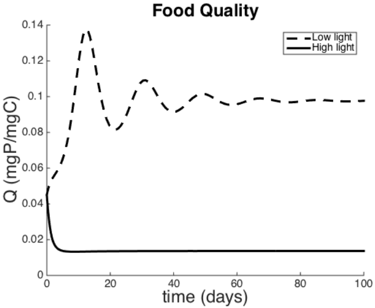
$$\Omega = \left\{ (x, y) : 0 \leq x \leq \mathbf{k} = \min \left\{ K, \frac{P}{q} \right\}, 0 \leq y, qx + \theta y < P \right\}$$

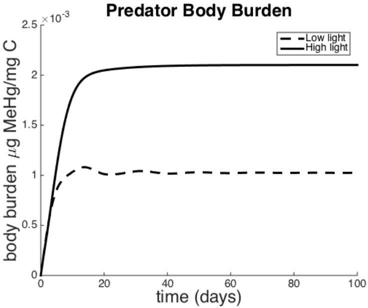
will remain there for all forward time.

Boundary equilibria

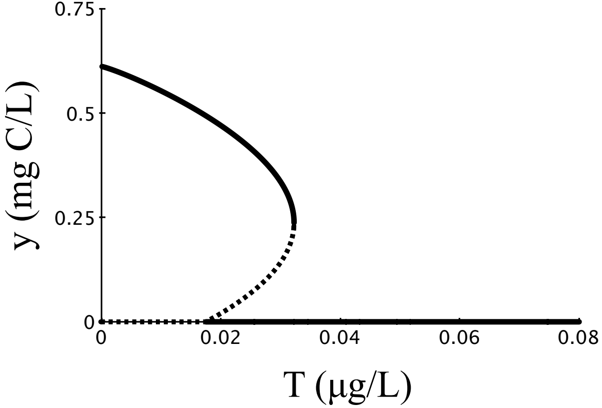
$E_0 = (0, 0)$ is saddle point. The stability of $E_1 = (\mathbf{k}, 0)$ depends on the sign of $G(\mathbf{k}, 0)$. E_1 is locally asymptotically stable if $G(\mathbf{k}, 0) < 0$ and E_1 is saddle point if $G(\mathbf{k}, 0) > 0$.

Model simulations

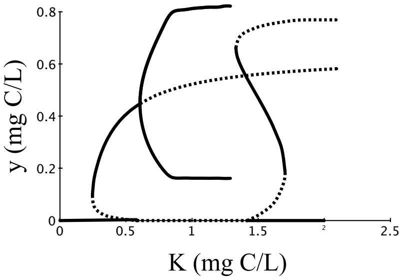


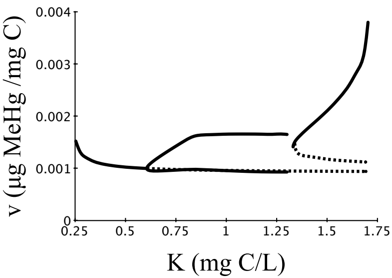


Bifurcation dynamics



Bifurcation dynamics





Conclusion

- Developed a predator-prey model of MeHg accumulation under vary nutrient constraints
- Parameterized the model for algae-*Daphnia* system
- Model predicts that rapid growth from high-quality food can reduce the accumulation and trophic transfer of MeHg in predators
- Ecological Stoichiometry can help improve testing protocols for assessing risk of exposures

Thank you

Banach Center
INSTITUTE OF MATHEMATICS PAS

Warsaw Center
of Mathematics
and Computer Science

UNIVERSITAS VARSOVIENSIS

icm

NSF

Society for
Mathematical
Biology