Rigorous derivation of nonlinear scalar conservation laws from follow-the-leader type models via many particle limit

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Micro and Macro Systems in Life Sciences
Będlewo, 8-12 June 2015

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Modelling traffic flow.

Practical goals.

- Rational planning and management of vehicle fluxes.
- Reduce environmental pollution and cities congestion.

Mathematical modelling.

- **Microscopic approach.** Each car is a ‘moving particle’ satisfying an ODE.
- **Macroscopic approach.** Averaged quantities satisfying PDEs.

Advantages of the macroscopic approach.

- Very powerful description of queues tails in terms of shocks.
- Suitable with very large number of vehicles.
- Easy to validate and implement (low number of parameters).
- Suitable to *real time* prediction, estimation, optimization and control.

Modelling traffic flow.

Practical goals.

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Mathematical modelling.

- **Microscopic approach.** Each car is a ‘moving particle’ satisfying an ODE.
- **Macroscopic approach.** Averaged quantities satisfying PDEs.

Continuum hypothesis is not satisfied!

The number of cars is far lower than that of molecules, for example, in gas dynamics (1 mole of gas contains 6×10^{23} molecules), the continuum assumption is not justified and the macroscopic formulation is not a priori justified!

Link between macroscopic and microscopic approach.

It provides a validation of the macroscopic approach and of the use of data collection from GPS devises when the number of detected ‘reference cars’ is very large.

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The macroscopic variables.

The macroscopic variables are:

ρ	density	#cars per unit length of the road
v	velocity	space covered per unit time by the cars
f	flow	#cars per unit time

The macroscopic variables satisfy:

by definition $\boxed{f = \rho v}$

by conservation of #cars $\boxed{\rho_t + f_x = 0}$

We have 3 variables and 2 equations!

Two macroscopic approaches.

- **First order** models close the system by giving an explicit expression of 1 of the 3 unknowns in terms of the remaining 2 (equation of state).
  Example: Lighthill-Whitham-Richards (1955, 1956).

$$\boxed{\rho_t + [\rho v(\rho)]_x = 0.}$$

  - Equilibrium models.
  - Velocity as function of the density.

- **Second order** models close the system by adding a further PDE.
  Example: Aw-Rascle-Zhang (1999, 2002).

$$\boxed{\rho_t + (\rho v)_x = 0, \qquad [v + p(\rho)]_t + v[v + p(\rho)]_x = 0.}$$

  - Continuum analogue of Newton's law.
  - Velocity evolves via a separate PDE.

Justification of the macroscopic models.

- **A posteriori**. Descriptive power (no physical laws).
- **A priori**. Validation via microscopic models (no continuum hypothesis).

The LWR model.

$$\boxed{\rho_t + [\rho v(\rho)]_x = 0, \qquad \rho(t=0) = \bar{\rho}}$$

Relevant parameters.

- Maximum density of vehicles $\rho_{\max} > 0$. We normalize $\rho_{\max} = 1$.
- Maximum possible speed $v_{\max} > 0$.
- Total length of the vehicles (constant in time)

$$L = \int_{\mathbb{R}} \rho(t,x) \mathrm{d}x > 0.$$

Main assumptions on the velocity map v.

- $v \in \mathbf{C}^1([0,1]; [0, v_{\max}])$.
- v strictly decreasing on $[0,1]$.
- $v(0) = v_{\max}$, $v(1) = 0$.

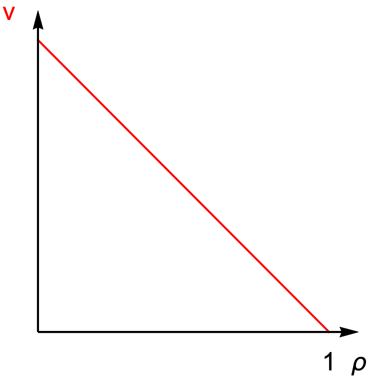
Initial condition.

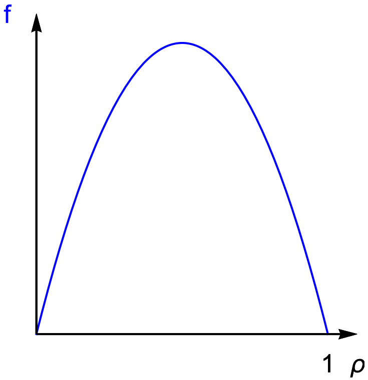
- $\rho(t=0) = \bar{\rho} \in \mathbf{L}^\infty(\mathbb{R})$, $\bar{\rho} \geq 0$, $\bar{\rho}$ with compact support.

Examples of velocities (from empirical observations).

Greenshields 1935:

$$\boxed{v(\rho) = v_{\max}(1-\rho)}$$

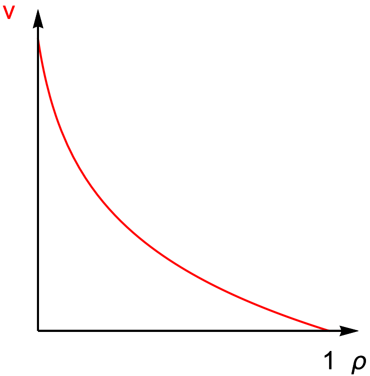


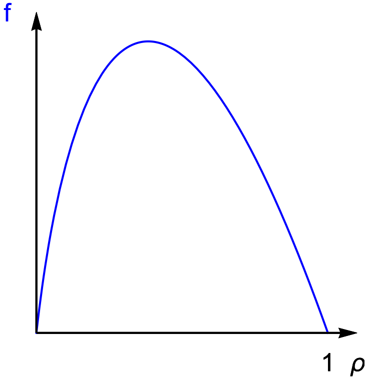


Examples of velocities (from empirical observations).

Greenberg 1959, ‘renormalized’ version

$$\boxed{v(\rho) = v_{\max} \left[\log\left(\frac{1+\alpha}{\alpha}\right)\right]^{-1} \log\left(\frac{1+\alpha}{\rho+\alpha}\right)} \qquad \alpha > 0$$



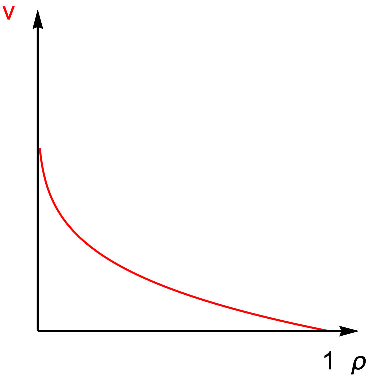


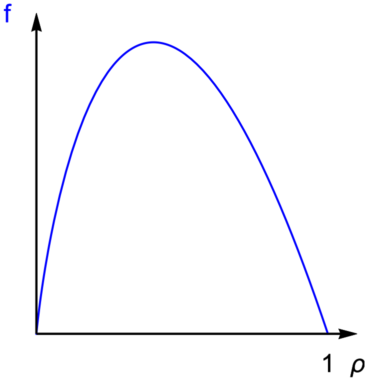
Examples of velocities (from empirical observations).

Pipes-Munjal 1967

$$\boxed{v(\rho) = v_{\max}\ (1 - \rho^{\alpha})} \qquad \alpha > 0$$

Example: $\alpha = 0.2$



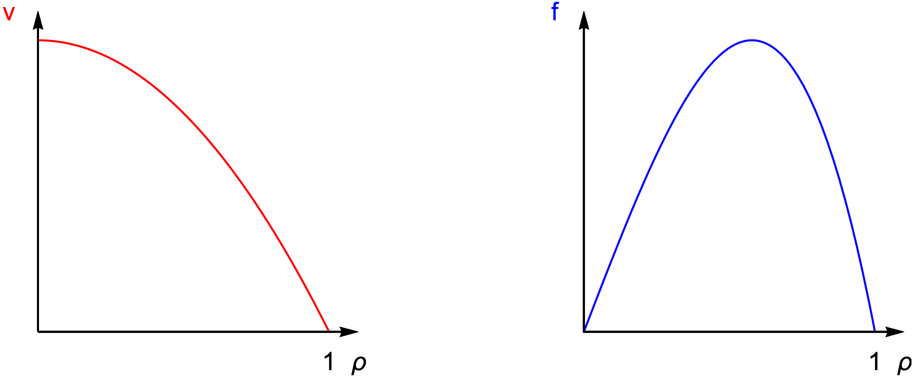


Examples of velocities (from empirical observations).

Pipes-Munjal 1967

$$\boxed{v(\rho) = v_{\max}\,(1-\rho^{\alpha})} \qquad \alpha > 0$$

Example: $\alpha = 2$



A quick review on the mathematical theory.

- Discontinuous solutions, weak solutions: for all $\phi \in \mathbf{C_c^1}([0,+\infty) \times \mathbb{R})$,

$$\int_{\mathbb{R}} \int_{\mathbb{R}_+} \Big[\rho(t,x)\,\phi_t(t,x) + f\left(\rho(t,x)\right)\phi_x(t,x)\Big]\,\mathrm{d}t\,\mathrm{d}x + \int_{\mathbb{R}} \bar\rho(x)\,\phi(0,x)\,\mathrm{d}x = 0$$

- Non uniqueness of weak solutions. Uniqueness of *entropy solutions* (Oleinik 1963, Kružkov 1970): for all test functions $\phi \geq 0$ and for all $k \in \mathbb{R}$,

$$\int_{\mathbb{R}} \int_{\mathbb{R}_+} \Big[|\rho(t,x)-k|\,\phi_t(t,x) + \mathrm{sgn}\left(\rho(t,x)-k\right)\left[f\left(\rho(t,x)\right)-f(k)\right]\phi_x(t,x)\Big]\,\mathrm{d}t\,\mathrm{d}x$$
$$+ \int_{\mathbb{R}} \phi(0,x)|\bar\rho(x)-k|\,\mathrm{d}x \geq 0 \qquad (1)$$

- Initial trace. Uniqueness if (1) is satisfied with $\phi(t=0)=0$ and initial trace is reached in the weak-$*$ $\mathbf{L}^\infty$-topology, provided f is not affine a.e. (Chen-Rascle 2000).
- *Oleinik condition*. Entropy solutions are characterised by

$$f'(\rho)_x \leq \frac{1}{t}, \qquad \text{in } \mathcal{D}'((0,+\infty)\times\mathbb{R}).$$

Constructing entropy solutions.

Regularization strategy.

- Vanishing viscosity, see Dafermos 2000 and the references therein.

Numerical methods.

- Finite differences. Glimm 1965.
- Wave front tracking method. Dafermos 1972.

Mesoscopic approximation.

- Kinetic approximation. Lions-Perthame-Tadmor 1994.

Microscopic probabilistic approach.

- Exclusion processes (list incomplete!). Rost 1982, Ferrari-Fouque 1987.

Microscopic deterministic approach.

- Sticky particles. Brenier-Grenier 1998.
- Lagrangian, follow-the-leader type systems.

  Formal derivation: Whitham 1974, Colombo-Rossi 2013.

  Rigorous derivation: Di Francesco-Rosini 2015.

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The follow-the-leader particle approximation.

Fix the initial condition $\bar{\rho}$ and let L be its total mass.

- Fix $n \in \mathbb{N}$ and let $\ell = \ell_n = L\,2^{-n}$ be the length of each platoon of cars.
- Consider $N + 1 = N_n + 1 = 2^n + 1$ (ordered) *reference cars* $x_0, x_1, \ldots, x_N$ corresponding to the end points of the N platoons.
- The cars $x_0, \ldots, x_N$ evolve according to

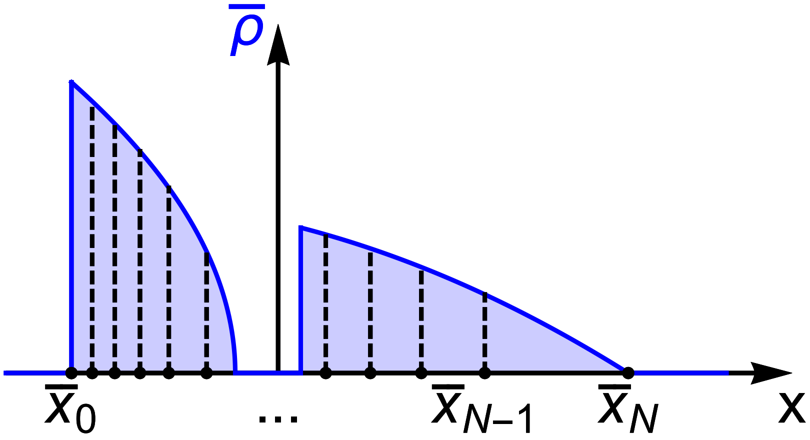
$$\dot{x}_N(t) = v_{\max},$$
$$\dot{x}_i(t) = v\left(\frac{\ell}{x_{i+1}(t) - x_i(t)}\right), \qquad i = N-1, \ldots, 0.$$

- The initial conditions of the cars are taken by atomization of $\bar{\rho}$, i.e.

$$x_0(t=0) = \bar{x}_0 = \min(\mathrm{spt}(\bar{\rho})),$$
$$x_i(t=0) = \bar{x}_i = \sup\left\{x \in \mathbb{R} \colon \int_{\bar{x}_{i-1}}^{x} \bar{\rho}(y)\,\mathrm{d}y < \ell\right\}, \qquad i = 1, \ldots, N.$$

Atomization of the initial condition.

The initial condition $\bar{\rho}$ is split into N parts with equal integral ℓ.



Large particle limit.

Empirical measure

$$\tilde{\rho}^n(t) = \sum_{i=1}^{N} \ell_n \, \delta_{x_i(t)}$$

Discrete density

$$\hat{\rho}^n(t,x) = \sum_{i=1}^{N} y_i^n(t) \chi_{[x_i(t), x_{i+1}(t))}(x), \qquad y_i^n(t) = \frac{\ell_n}{x_{i+1}(t) - x_i(t)}$$

Goal:
Prove that $\tilde{\rho}^n(t)$ and $\hat{\rho}^n(t,\cdot)$ converge to the unique entropy solution ρ of the LWR equation with $\bar{\rho}$ as initial condition.

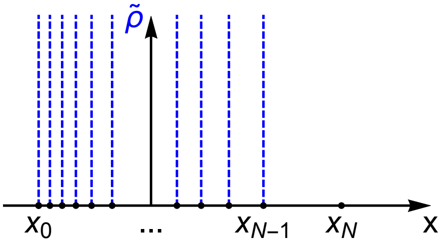
Empirical measure and discrete density.

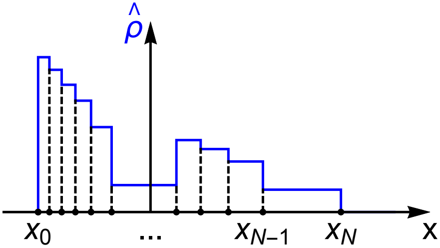
Empirical measure

$$\tilde{\rho}(t) = \sum_{i=1}^{N} \ell \, \delta_{x_i(t)}$$

Discrete density

$$\hat{\rho}(t,x) = \sum_{i=1}^{N} y_i(t) \chi_{[x_i(t), x_{i+1}(t))}(x)$$





Heuristic derivation.

- Let ρ be the entropy solution of

$$\rho_t + [\rho v(\rho)]_x = 0. \tag{2}$$

- Let F be the cumulative distribution of ρ:

$$F(x,t) = \int_{-\infty}^{x} \rho(t,y)\mathrm{d}y.$$

- Let X be the pseudo inverse of F:

$$X(t,z) = \inf\{x \in \mathbb{R} \colon F(x) > z\}, \qquad z \in [0,L].$$

- Formally, $X(t,z)$ satisfies $F(t,X(t,z)) = z$ and

$$\left.\begin{array}{l} F_x = \rho \\ F_t = -\rho v(\rho) \end{array}\right\} \Rightarrow \left.\begin{array}{l} 1 = F(t,X(t,z))_z = F_x X_z = \rho X_z \\ 0 = F(t,X(t,z))_t = F_t + F_x X_t = \rho(X_t - v(\rho)) \end{array}\right\} \Rightarrow X_t = v\left(\frac{1}{X_z}\right). \tag{3}$$

- Forward z-finite difference of (3) with step ℓ gives

$$X_t(t,z) = v\left(\frac{\ell}{X(t,z+\ell) - X(t,z)}\right), \qquad z = 0,\ldots,(N-1)\ell. \tag{4}$$

The follow-the-leader system (4) is the discrete Lagrangian version of (2).

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Convergence Theorem.

Let ρ be the unique entropy solution of

$$\rho_t + [\rho v(\rho)]_x = 0, \qquad \rho(t=0) = \bar{\rho},$$

where $\bar{\rho}$ is in $\mathcal{M}_L \cap \mathbf{L}^\infty(\mathbb{R})$, v is in $\mathbf{C}^1(\mathbb{R}_+)$, strictly decreasing with $v(0) = v_{\max} > 0$.

Theorem (MDF and MDR, ARMA 2015.)

If

- $\bar{\rho} \in \mathbf{BV}(\mathbb{R})$,

or

- $\mathbb{R}_+ \ni \rho \mapsto \rho v'(\rho) \in \mathbb{R}_+$ *is non-increasing.*

Then,

- *the sequence* $\hat{\rho}^n \to \rho$ *almost everywhere and in* $\mathbf{L}^1_{\mathbf{loc}}([0,+\infty) \times \mathbb{R})$.
- *the sequence* $\tilde{\rho}^n \to \rho$ *in the topology* $\mathbf{L}^1_{\mathbf{loc}}([0,+\infty); d_{L,1})$, *where* $d_{L,1}$ *is the scaled* 1*-Wasserstein distance.*

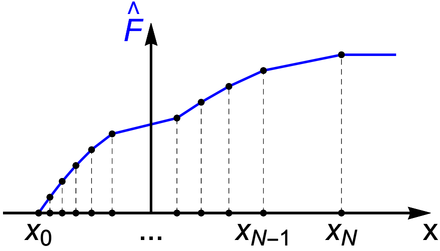
$$\mathcal{M}_L = \{\rho \text{ Radon measure on } \mathbb{R} \text{ with compact support} \colon \rho \geq 0,\ \rho(\mathbb{R}) = L\}$$

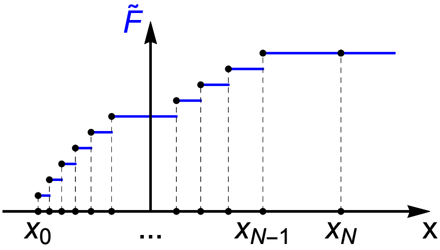
$$d_{L,1}(\rho_1, \rho_2) = L\, d_1(\rho_1/L, \rho_2/L) = \|F_{\rho_1} - F_{\rho_2}\|_{\mathbf{L}^1(\mathbb{R})}$$

Ingredients: cumulative distributions.

$x \mapsto \hat{F}^n(t,x) = \int_{-\infty}^{x} \hat{\rho}^n(t,y)\mathrm{d}y$ is **PLC**

$x \mapsto \tilde{F}^n(t,x) = \tilde{\rho}^n(t)((-\infty,x])$ is **PC**

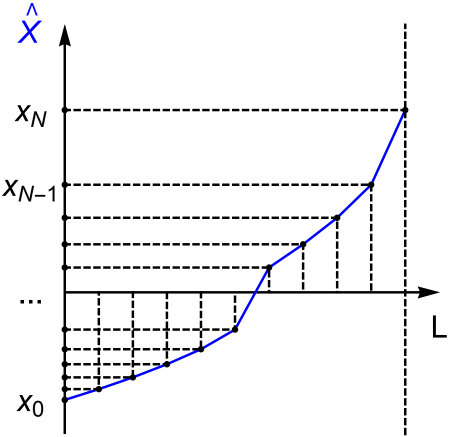


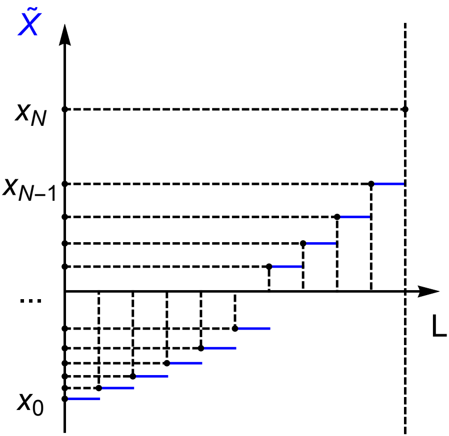


Ingredients: pseudo inverses.

$\hat{X}^n$ pseudo-inverse of $\hat{F}^n$ is **PLC**

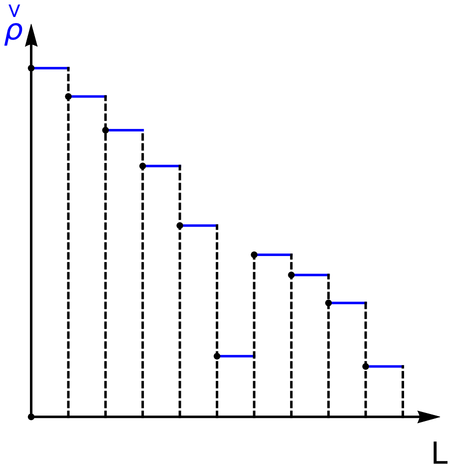
$\tilde{X}^n$ pseudo-inverse of $\tilde{F}^n$ is **PC**





Ingredients: discrete Lagrangian density.

$$\check{\rho}^n = \hat{\rho}^n \circ \hat{X}^n \text{ is } \textbf{PC}$$



Strategy of the proof.

(i)
- Prove that $(\tilde{X}^n)_n$ has a strong limit X in $\mathbf{L}^1_{\mathbf{loc}}([0,+\infty[\times[0,L])$, equivalent to $(\tilde{\rho}^n)_n$ converging to a measure ρ in $\mathbf{L}^1_{\mathbf{loc}}([0,+\infty[;\, d_{L,1})$.
- Prove that $(\hat{X}^n)_n$ converges in $\mathbf{L}^1_{\mathbf{loc}}([0,+\infty[\times[0,L])$ to the same limit X, i.e. $(\hat{\rho}^n)_n$ converges to ρ in $\mathbf{L}^1_{\mathbf{loc}}([0,+\infty[;\, d_{L,1})$.

(ii)
- Prove that X has difference quotients bounded below by 1, i.e. ρ is actually in $\mathbf{L}^\infty$ and is a.e. bounded by 1.
- This easily implies weak-$*$ convergence of $(\check{\rho}^n)_n$ to a limit $\check{\rho}$ in $\mathbf{L}^\infty$.
- It remains to prove that $\check{\rho} \circ F = \rho$, and that such limit is the unique entropy solution to LWR. This requires stronger estimates on $\hat{\rho}^n$.

(iii)
- Case $\bar{\rho} \in \mathbf{BV}$: direct uniform **BV** estimate of $\hat{\rho}^n$.
- Case $\bar{\rho} \in \mathbf{L}^\infty$+additional assumption on v: uniform *discrete Oleinik condition* for $\check{\rho}^n$, which implies uniform **BV** estimate for $\check{\rho}^n$, i.e.for $\hat{\rho}^n$.

(iv) Prove that ρ is a weak solution: it follows from letting $n \to +\infty$ in the formulation of the FTL system

$$\tilde{X}^n_t = v(\check{\rho}^n).$$

(v) Prove that ρ is an entropy solution: in the discrete setting, use strong $\mathbf{L^1}$ compactness to pass it to the limit.

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Discrete maximum principle.

The global existence for the FTL system is guaranteed by the following

Lemma (Discrete maximum principle)

For all $i = 0, \ldots, N-1$, we have $\ell \leq x_{i+1}(t) - x_i(t)$ for all times $t \geq 0$.

Proof.

- Replace v with its extension $V = v\chi_{[0,1]}$, V is still Lipschitz.
- Assume by contradiction $x_{i+1}(t_1) - x_i(t_1) = \ell$, and $x_{i+1}(t) - x_i(t) < \ell$ on $t \in (t_1, t_2]$.
- Integrate FTL on $[t_1, t]$:

$$x_i(t) = x_i(t_1) + \int_{t_1}^{t} V\left(\frac{\ell}{x_{i+1}(\tau) - x_i(\tau)}\right) \mathrm{d}\tau = x_i(t_1).$$

- $\ell > x_{i+1}(t) - x_i(t) \geq x_{i+1}(t_1) - x_i(t_1) = \ell$, contradiction!
- By uniqueness, the same holds for the system with v.

$\square$

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Strong compactness of $\hat{X}^n$ and $\tilde{X}^n$.

1-dimensional Wasserstein distance

$$\left.\begin{array}{l} F_i \text{ cumulative distribution of } \rho_i \\ X_i \text{ pseudo-inverse of } F_i \end{array}\right\} \Rightarrow d_{L,1}(\rho_1, \rho_2) = \|X_1 - X_2\|_{\mathbf{L^1}([0,L])}$$

Proposition

There exists a unique $X \in \mathbf{L}^\infty(\mathbb{R}_+ \times [0,L])$, monotone non-decreasing and right continuous with respect to z, such that

$$(\hat{X}^n)_n \text{ and } (\tilde{X}^n)_n \text{ converge to } X \text{ in } \mathbf{L}^1_{\mathbf{loc}}(\mathbb{R}_+ \times [0,L]),$$

and for any $t, s > 0$

$$\mathrm{TV}\,[X(t)] \le |\bar{x}_N - \bar{x}_0 + v_{\max}\, t|, \tag{5a}$$

$$\|X(t)\|_{\mathbf{L}^\infty([0,L];\mathbb{R})} \le \max\{|\bar{x}_0|, |\bar{x}_N + v_{\max}\, t|\}, \tag{5b}$$

$$\int_0^L |X(t,z) - X(s,z)|\, \mathrm{d}z \le v_{\max}\, L\, |t - s|. \tag{5c}$$

Moreover, $(\tilde{X}^n)_n$ converges to X a.e. on $\mathbb{R}_+ \times [0,L]$.

Strong compactness of $\hat{X}^n$ and $\tilde{X}^n$.

Proof.

Fix $T > t > s \geq 0$. Estimates (5a) and (5b) are immediately proven for $\tilde{X}^n$.

$$\int_0^L \left|\tilde{X}^n(t,z) - \tilde{X}^n(s,z)\right| \mathrm{d}z = \sum_{i=0}^{N_n-1} \ell_n \left[x_i^n(t) - x_i^n(s)\right]$$

$$= \sum_{i=0}^{N_n-1} \ell_n \left[\int_s^t v\left(y_i^n(\tau)\right) \mathrm{d}\tau\right] \leq v_{\max} L\,(t-s)\,.$$

By Helly's theorem, $\tilde{X}^n$ converges strongly as in the statement up to a subsequence. As $\tilde{X}^n$ is monotone in n, the whole sequence converges to a unique limit X.

$$\int_0^L \left|\hat{X}^n(t,z) - \tilde{X}^n(t,z)\right| \mathrm{d}z = \sum_{i=0}^{N_n-1} y_i^n(t)^{-1} \int_{i\,\ell_n}^{(i+1)\,\ell_n} \left[z - i\,\ell_n\right] \mathrm{d}z$$

$$= \frac{\ell_n}{2} \sum_{i=0}^{N_n-1} \left[x_{i+1}^n(t) - x_i^n(t)\right] = \frac{\ell_n}{2} \left[x_{N_n}^n(t) - x_0^n(t)\right] \leq \frac{\ell_n}{2} \left[\bar{x}_{\max} - \bar{x}_{\min} + v_{\max}\,T\right],$$

Hence, $\hat{X}^n$ and $\tilde{X}^n$ have the same limit. $\square$

$\mathbf{L}^\infty$ bound for the limit measure.

- Let $0 \leq z_1 < z_2 \leq L$.
- For n sufficiently large, let $i\,\ell_n \leq z_1 < (i+1)\,\ell_n$ and $\ell_n\,j \leq z_2 < \ell_n\,(j+1)$
- The discrete maximum principle implies

$$\frac{\tilde{X}^n(t,z_2) - \tilde{X}^n(t,z_1)}{z_2 - z_1} \geq \frac{x_j^n(t) - x_i^n(t)}{(j+1)\,\ell_n - i\,\ell_n} \geq \frac{(j-i)\,\ell_n}{(j+1)\,\ell_n - i\,\ell_n}$$
$$= 1 - \frac{1}{j-i+1} \geq 1 - \frac{1}{(z_2\,\ell_n^{-1} - 1) - z_1\,\ell_n^{-1} + 1} = 1 - \frac{\ell_n}{z_2 - z_1}.$$

- By sending $n \to +\infty$, we get

$$\partial_z X(t,\cdot) \geq 1 \qquad \text{in } \mathcal{D}'.$$

- Let $F(t,\cdot)$ be the pseudo inverse of $X(t,\cdot)$. Then $\rho = F_x$ satisfies

$$\rho(t,x) \leq 1.$$

- $\tilde{\rho}^n$ and $\hat{\rho}^n$ converge to ρ in $\mathbf{L}^1_{\mathbf{loc}}([0,+\infty);\ d_{L,1})$.
- $\check{\rho}^n = \hat{\rho}^n \circ \hat{X}^n$ converges to some limit $\check{\rho}$ up to a subsequence in the weak-$*$ $\mathbf{L}^\infty$ topology.

A discrete Lagrangian version of LWR.

In the sequel, we shall make an extensive use of the discrete equations

$$\begin{aligned}
\dot{y}_i^n(t) &= -\frac{y_i^n(t)^2}{\ell_n}\Big[v(y_{i+1}^n(t)) - v(y_i^n(t))\Big], && i = 0, \ldots, N-2,\\
\dot{y}_{N-1}^n(t) &= -\frac{y_{N-1}^n(t)^2}{\ell_n}\Big[v_{\max} - v(y_{N-1}^n(t))\Big].
\end{aligned}$$

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BV contraction for **BV** initial data.

Proposition (**BV** contractivity for **BV** initial data)

Assume $\bar\rho \in \mathbf{BV}$. *Then for any* $n \in \mathbb{N}$

$$\mathrm{TV}\left[\hat\rho^n(t)\right] = \mathrm{TV}\left[\check\rho^n(t)\right] \le \mathrm{TV}\left[\bar\rho\right] \qquad \textit{for all } t \ge 0.$$

Proof. (We omit the index n for simplicity)

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{TV}\left[\hat\rho(t)\right] &= \frac{\mathrm{d}}{\mathrm{d}t}\left[y_0 + y_{N-1} + \sum_{i=0}^{N-2}|y_i - y_{i+1}|\right] \\
&= \dot y_0 + \dot y_{N-1} + \sum_{i=0}^{N-2}\mathrm{sgn}\left[y_i - y_{i+1}\right]\left[\dot y_i - \dot y_{i+1}\right] \\
&= \Big[1 + \mathrm{sgn}\left[y_0 - y_1\right]\Big]\,\dot y_0 + \Big[1 - \mathrm{sgn}\left[y_{N-2} - y_{N-1}\right]\Big]\,\dot y_{N-1} \\
&\quad + \sum_{i=1}^{N-2}\Big[\mathrm{sgn}\left[y_i - y_{i+1}\right] - \mathrm{sgn}\left[y_{i-1} - y_i\right]\Big]\,\dot y_i.
\end{aligned}$$

BV contraction for BV initial data.

Proof (continued): We analise all the terms above, and we get

$$\Big[1+\operatorname{sgn}[y_0-y_1]\Big]\,\dot{y}_0 = -\Big[1+\operatorname{sgn}[y_0-y_1]\Big]\,\frac{y_0^2}{\ell}\,[v(y_1)-v(y_0)] \le 0,$$

$$\Big[1-\operatorname{sgn}[y_{N-2}-y_{N-1}]\Big]\,\dot{y}_{N-1} = -\Big[1-\operatorname{sgn}[y_{N-2}-y_{N-1}]\Big]\,\frac{y_{N-1}^2}{\ell}\,[v_{\max}-v(y_{N-1})] \le 0,$$

$$\Big[\operatorname{sgn}[y_i-y_{i+1}]-\operatorname{sgn}[y_{i-1}-y_i]\Big]\,\dot{y}_i =$$

$$= -\Big[\operatorname{sgn}[y_i-y_{i+1}]-\operatorname{sgn}[y_{i-1}-y_i]\Big]\,\frac{y_i^2}{\ell}\,[v(y_{i+1})-v(y_i)] \le 0.$$

Therefore, $\mathrm{TV}[\hat{\rho}(t)] \le \mathrm{TV}[\bar{\rho}]$ for all $t \ge 0$.

Discrete Oleinik condition.

Lemma (Discrete Oleinik-type condition)

Assume v satisfies the additional assumption $\rho v'(\rho)$ non-increasing. Then, for any $i = 0, \ldots, N_n - 2$ we have

$$t\, y_i^n(t) \left[v\left(y_{i+1}^n(t)\right) - v\left(y_i^n(t)\right) \right] \leq \ell_n \qquad \text{for all } t \geq 0. \tag{6}$$

Remark

Condition (6) in terms of $x_i(t)$

$$\frac{v\left(y_{i+1}^n(t)\right) - v\left(y_i^n(t)\right)}{x_{i+1}(t) - x_i(t)} \leq \frac{1}{t} \qquad \text{for all } t \geq 0. \tag{7}$$

(7) is a discrete analogous of

$$v(\rho)_x \leq \frac{1}{t}.$$

However, the sharp form of the Oleinik condition for the scalar conservation law is (cf. Hoff 1983)

$$f'(\rho)_x = (v(\rho) + \rho v'(\rho))_x \leq \frac{1}{t}.$$

Proof of the discrete Oleinik condition.

Notation: (we omit the dependence on n and t)

$$\begin{aligned} z_i &\doteq t\, y_i\, [v(y_{i+1}) - v(y_i)], && i = 0, \ldots, N-2, \\ z_{N-1} &\doteq t\, y_{N-1}\, [v_{\max} - v(y_{N-1})]. \end{aligned}$$

The Lemma is proven once we provide the estimate $z_i \leq \ell$ for $i = 1, \ldots, N-1$.

- **STEP 0: $\mathbf{z_{N-1}} \leq \boldsymbol{\ell}$.**

$$\begin{aligned} \dot{z}_{N-1} &= y_{N-1}\,[v_{\max} - v(y_{N-1})] + t\,\dot{y}_{N-1}\,[v_{\max} - v(y_{N-1})] - t\, y_{N-1}\, v'(y_{N-1})\,\dot{y}_{N-1} \\ &= y_{N-1}\,[v_{\max} - v(y_{N-1})] - \frac{t\, y_{N-1}^2}{\ell}\,[v_{\max} - v(y_{N-1})]^2 \\ &\quad + \frac{t\, v'(y_{N-1})\, y_{N-1}^3}{\ell}\,[v_{\max} - v(y_{N-1})] \\ &\leq y_{N-1}\,[v_{\max} - v(y_{N-1})]\left[1 - \frac{z_{N-1}}{\ell}\right]. \end{aligned}$$

Since $z_{N-1}(0) = 0$, from the above estimate we get $z_{N-1}(t) \leq \ell$ for all $t \geq 0$.

Proof of the discrete Oleinik condition (continued).

- **STEP 1: $\mathbf{z_{i+1} \leq \ell \Rightarrow z_i \leq \ell}$.**

$$
\begin{aligned}
\dot{z}_i &= y_i\,[v(y_{i+1}) - v(y_i)] + t\,\dot{y}_i\,[v(y_{i+1}) - v(y_i)] + t\,y_i\left[v'(y_{i+1})\,\dot{y}_{i+1} - v'(y_i)\,\dot{y}_i\right]\\
&= y_i\,[v(y_{i+1}) - v(y_i)] - \frac{t\,y_i^2}{\ell}\,[v(y_{i+1}) - v(y_i)]^2\\
&\quad + t\,y_i\left[-\frac{v'(y_{i+1})\,y_{i+1}^2}{\ell}\,[v(y_{i+2}) - v(y_{i+1})] + \frac{v'(y_i)\,y_i^2}{\ell}\,[v(y_{i+1}) - v(y_i)]\right]\\
&= y_i\,[v(y_{i+1}) - v(y_i)] - \frac{y_i}{\ell}\,[v(y_{i+1}) - v(y_i)]\,z_i - \frac{v'(y_{i+1})\,y_i\,y_{i+1}}{\ell}\,z_{i+1} + \frac{v'(y_i)\,y_i^2}{\ell}\,z_i.
\end{aligned}
$$

Since $\operatorname{sgn}_+[z_i] = \operatorname{sgn}_+[v(y_{i+1}) - v(y_i)] = \operatorname{sgn}_+[y_i - y_{i+1}]$ for all $t > 0$, from the assumption on z_{i+1} we easily obtain

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}[z_i]_+ &= y_i\,[v(y_{i+1}) - v(y_i)]_+ - \frac{y_i}{\ell}\,[v(y_{i+1}) - v(y_i)]_+\,[z_i]_+\\
&\quad - \frac{v'(y_{i+1})\,y_i\,y_{i+1}}{\ell}\,\operatorname{sgn}_+[z_i]\,z_{i+1} + \frac{v'(y_i)\,y_i^2}{\ell}\,[z_i]_+\\
&\leq y_i\,[v(y_{i+1}) - v(y_i)]_+\left[1 - \frac{[z_i]_+}{\ell}\right] - v'(y_{i+1})\,y_i\,y_{i+1}\,\operatorname{sgn}_+[z_i] + \frac{v'(y_i)\,y_i^2}{\ell}\,[z_i]_+.
\end{aligned}
$$

Proof of the discrete Oleinik condition (continued).

The additional condition on v gives $-v'(y_{i+1})y_{i+1} \leq -v'(y_i)y_i$ for $y_i \geq y_{i+1}$, and then

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}[z_i]_+ &\leq y_i\,[v(y_{i+1}) - v(y_i)]_+ \left[1 - \frac{[z_i]_+}{\ell}\right] - v'(y_i)\,y_i^2\,\mathrm{sgn}_+[z_i] + \frac{v'(y_i)\,y_i^2}{\ell}\,[z_i]_+ \\ &= y_i \left[[v(y_{i+1}) - v(y_i)]_+ - v'(y_i)\,y_i\right]\left[1 - \frac{[z_i]_+}{\ell}\right].\end{aligned}$$

Now, as $v' \leq 0$, and since $z_i(0) = 0$, we get that $z_i(t)_+ \leq \ell$ for all $t \geq 0$.

• **STEP 2: $\mathbf{z_{N-2}} \leq \ell$.** From analogous computations as in previous step, by using the monotonicity of $y \mapsto y\,v'(y)$ and Step 0, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}[z_{N-2}]_+ \leq y_{N-2}\left[[v(y_{N-1}) - v(y_{N-2})]_+ - v'(y_{N-2})\,y_{N-2}\right]\left[1 - \frac{[z_{N-2}]_+}{\ell}\right].$$

Again, $v' \leq 0$ and $z_{N-2}(0) = 0$ imply that $z_{N-2}(t)_+ \leq \ell$ for all $t \geq 0$.

• **CONCLUSION**. The estimate (6) is proven recursively: Step 2 provides the first step with $i = N-2$, whereas Step 1 proves that the estimate holds for all $i \in \{0, \ldots, N-3\}$.

Remark: The discrete Oleinik condition provides a uniform **BV** estimate away from $t = 0$. Here we use that solutions have compact support.

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A technical problem.

Typically (e.g. the wave-front-tracking algorithm for conservation laws) $\mathbf{L^1}$-continuity of the approximating sequence gives the desired compactness via Helly's Theorem. Here *we are able to prove such an estimate for* $\check{\rho}^n$*, but not for* $\hat{\rho}^n$*.*

Proposition (Uniform $\mathbf{L^1}$-continuity in time of $\check{\rho}^n$)

For any $\delta > 0$ *we have*

$$\int_0^L \left|\check{\rho}^n(t,z) - \check{\rho}^n(s,z)\right| \mathrm{d}z \leq C|t-s| \qquad \text{for all } t, s \geq \delta,$$

with some C *depending on* δ*.*

Proof.

(Sketched) A direct computation of the l.h.s. and the discrete maximum principle give

$$\int_0^L \left|\check{\rho}^n(t,z) - \check{\rho}^n(s,z)\right| \mathrm{d}z = \sum_{i=0}^{N_n-1} \ell_n \left|y_i^n(t) - y_i^n(s)\right| \leq \int_s^t \left[\mathrm{TV}\left[v\left(\check{\rho}^n(\tau)\right)\right] + v_{\max}\right] \mathrm{d}\tau.$$

□

How to solve the technical problem?

Proposition (Uniform Wasserstein time continuity of $\hat{\rho}^n$)

For any $n \in \mathbb{N}$ we have

$$d_{L,1}\left(\hat{\rho}^n(t), \hat{\rho}^n(s)\right) \leq 2\, L\, v_{\max}\, |t-s| \qquad \textit{for all } s, t \geq 0.$$

Proof.

$$d_{L,1}\left(\hat{\rho}^n(t), \hat{\rho}^n(s)\right) = \left\| \hat{X}^n(t) - \hat{X}^n(s) \right\|_{\mathbf{L}^1([0,L];\mathbb{R})} = \sum_{i=0}^{N_n-1} \int_{i\,\ell_n}^{(i+1)\,\ell_n} \left[\hat{X}^n(t,z) - \hat{X}^n(s,z)\right] \mathrm{d}z$$

$$= \sum_{i=0}^{N_n-1} \ell_n \left[x_i^n(t) - x_i^n(s)\right] + \sum_{i=0}^{N_n-1} \left[y_i^n(t)^{-1} - y_i^n(s)^{-1}\right] \int_{i\,\ell_n}^{(i+1)\,\ell_n} (z - i\,\ell_n)\, \mathrm{d}z$$

$$= \sum_{i=0}^{N_n-1} \ell_n \int_s^t v\left(y_i^n(\tau)\right) \mathrm{d}\tau + \sum_{i=0}^{N_n-1} \frac{\ell_n^2}{2} \int_s^t \frac{\mathrm{d}}{\mathrm{d}\tau} \left[y_i^n(\tau)^{-1}\right] \mathrm{d}\tau$$

$$\leq L\, v_{\max}\, (t-s) + \frac{\ell_n}{2} \int_s^t \left[v_{\max} - v\left(y_0^n(\tau)\right)\right] \mathrm{d}\tau \leq 2\, L\, v_{\max}\, (t-s).$$

$\square$

A generalisation of Aubin-Lions Lemma.

The desired compactness then follows from the following

Theorem (Generalized Aubin-Lions lemma, Rossi-Savaré 2003)

Let T, $L > 0$ and $I \subset \mathbb{R}$ be a bounded open convex interval. Assume $w \colon \mathbb{R} \to \mathbb{R}$ is a Lipschitz continuous and strictly monotone function. Let $(\rho^n)_{n\in\mathbb{N}}$ be a nonnegative sequence in $\mathbf{L}^\infty\left((0,T) \times \mathbb{R}; \mathbb{R}\right)$, with compact support and fixed mass $L > 0$, such that:

- $\rho^n \colon (0,T) \to \mathbf{L^1}(\mathbb{R};\mathbb{R})$ *is measurable for all* $n \in \mathbb{N}$;
- $\mathrm{spt}\left(\rho^n(t)\right) \subseteq I$ *for all* $t \in \left]0,T\right[$ *and* $n \in \mathbb{N}$;
- $\displaystyle\sup_{n\in\mathbb{N}} \int_0^T \left[\left\|w\left(\rho^n(t)\right)\right\|_{\mathbf{L^1}(I;\mathbb{R})} + \mathrm{TV}\left[w\left(\rho^n(t)\right)\right]\right] \mathrm{d}t < +\infty$;
- *There exists a constant C depending only on T such that* $d_{L,1}\left(\rho^n(s), \rho^n(t)\right) \le C\,|t-s|$ *for all* $s, t \in \left]0,T\right[$ *and* $n \in \mathbb{N}$.

Then, $(\rho^n)_{n\in\mathbb{N}}$ is strongly relatively compact in $\mathbf{L^1}((0,T) \times \mathbb{R};\mathbb{R})$.

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Changing variable in the limit.

Another important property that we need to check is

$$\check{\rho}(t, F(t,x)) = \rho(t,x), \qquad \text{on } \mathrm{spt}(\rho),$$

where

- $\check{\rho}$ is the strong limit of $\check{\rho}^n$,
- ρ is the strong limit of $\hat{\rho}^n$,
- F is the cumulative distribution of ρ.

This is ensured by the strong compactness of both $\check{\rho}^n$ and $\hat{\rho}^n$.

Weak solutions in the limit.

Proposition

The limit function ρ of $\hat{\rho}^n$ is a weak solution of the LWR equation with i.c. $\bar{\rho}$.

Proof.

Let $\phi \in \mathbf{C}_\mathbf{c}^\infty\left([0,+\infty[\times \mathbb{R}; \mathbb{R}\right)$. We have

$$\int_{\mathbb{R}_+} \int_0^L \left[v\left(\check{\rho}^n(t,z)\right) \phi_x\left(t, \tilde{X}^n(t,z)\right) \right] \mathrm{d}z \mathrm{d}t = \sum_{i=0}^{N_n-1} \int_{\mathbb{R}_+} \int_{i\,\ell_n}^{(i+1)\,\ell_n} \left[v\left(y_i^n(t)\right) \phi_x\left(t, x_i^n(t)\right) \right] \mathrm{d}z\,\mathrm{d}t$$

$$= \sum_{i=0}^{N_n-1} \int_{\mathbb{R}_+} \int_{i\,\ell_n}^{(i+1)\,\ell_n} \left[\frac{\mathrm{d}}{\mathrm{d}t} \phi\left(t, x_i^n(t)\right) - \phi_t\left(t, x_i^n(t)\right) \right] \mathrm{d}z\,\mathrm{d}t$$

$$= -\int_0^L \phi\left(0, \tilde{X}^n(0,z)\right) \mathrm{d}z - \int_{\mathbb{R}_+} \int_0^L \phi_t\left(t, \tilde{X}^n(t,z)\right) \mathrm{d}z\,\mathrm{d}t.$$

By the strong convergence of $\tilde{X}^n$ and $\check{\rho}^n$, by chancing variable $x = X(t,z)$, and by using $\check{\rho}\left(t, F(t,x)\right) = \rho(t,x)$ a.e. on $\mathrm{spt}(\rho)$, we get the definition of weak solution for LWR. □

Entropy solutions in the limit.

Conclusion of the proof.

- We need to prove that the limit ρ is an entropy solution in the Kružkov sense.
- This is trivial in the uniformly concave case $f'' \leq -\epsilon < 0$, as in that case $f'(\rho) \simeq v(\rho)$, and one can obtain the sharp Oleinik condition $f'(\rho)_x \leq 1/t$ in the limit.
- In the general case, we need to use the definition of entropy solution by Kružkov. This follows from the inequality

$$\int_{\mathbb{R}_+}\int_{\mathbb{R}} \left[\left| \hat{\rho}^n(t,x) - k \right| \phi_t(t,x) + \mathrm{sgn}(\hat{\rho}^n(t,x) - k) \left[f(\hat{\rho}^n(t,x)) - f(k) \right] \phi_x(t,x) \right] \mathrm{d}x \, \mathrm{d}t \geq o(1),$$

as $n \to +\infty$, which is very technical and is omitted here.

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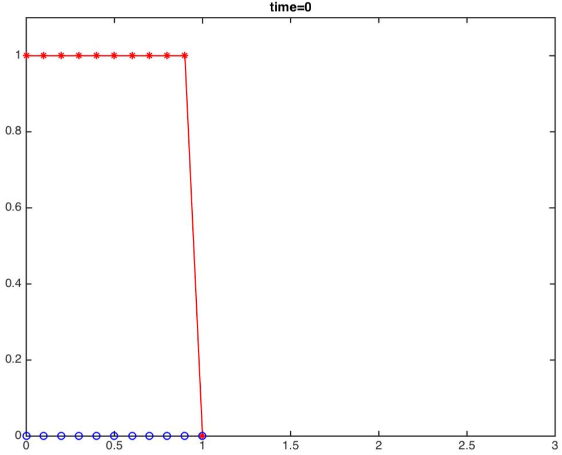
Concluding remarks.

- We remark that our set of assumptions on v allows for degenerate concave fluxes at zero.
- In the case of linear velocity v, e.g. $v(\rho) = v_{\max}(1-\rho)$, the convergence to a weak solution can be obtained without the need of the **BV** estimates, as the velocity term in the pseudo-inverse PDE is linear. This is somehow intrinsic in using a Lagrangian description.
- In order to get continuity in time for the sequence $\hat{\rho}^n$, the most natural try would be getting $\mathbf{L^1}$-continuity. Encouraged by the $\mathbf{L^1}$ time equi-continuity of $\check{\rho}^n$, we have attempted at proving such a property in many ways without success. This is the reason why use the generalized Aubin-Lions lemma, which allows to take advantage of the Wasserstein equi-continuity of $\hat{\rho}^n$, and still get the same $\mathbf{L^1}$-compactness in the end. The only drawback of this strategy is that we can't get any $\mathbf{L^1}$ time continuity for the limit.
- Our approach has the advantage of providing a piecewise constant approximation with a non increasing number of jumps. The price to pay for such a simplification is that *we lose the classical shock structure at a microscopic level*. Indeed, the explicit solution to the FTL system even for simple Riemann-type initial conditions is not immediate.

Future projects.

- Extending the results to Dirichlet boundary conditiont (phantom moving particles at the boundary).
- Use this strategy to attack the existence theory of similar models, e.g. with discontinuous flux.

Numerical simulation.



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