Towards an Epidemiological Model for the Spread of an Infectious Disease with Quarantine

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Standard SIR Model

$$\dot{S} = \nu N - \beta I S - \nu S - \kappa\omega S$$

$$\dot{I} = \beta I S - \nu I - \alpha\tau I$$

$$\dot{R} = \alpha\tau I + \kappa\omega S - \nu R$$

Assumptions:

- **constant population** $N = S + I + R = \text{const}$
- **no "deaths" from disease / deaths from disease are "removed"**

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Assumptions:

- **constant population** $N = S + I + R = \text{const}$
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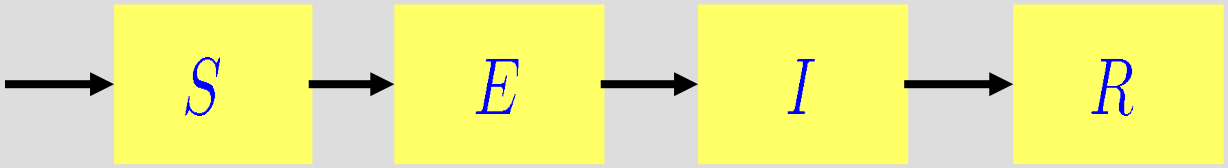
If there are significant deaths from the disease, this should be accounted for and population is really time varying

An SEIR Model with Time-Varying Populations



Sun and Hsieh, Appl Math Mod, 2010

$$\dot{S}(t) = \gamma N(t) - \beta \frac{I(t)}{N(t)} S(t) - \nu S(t) - \kappa\omega S(t)$$

$$\dot{E}(t) = \beta \frac{I(t)}{N(t)} S(t) - (\delta + \nu)\, E(t)$$

$$\dot{I}(t) = \delta E(t) - (\mu + \nu)\, I(t) - \alpha\tau I(t)$$

$$\dot{R}(t) = \alpha\tau I(t) + \kappa\omega S(t) - \nu R(t)$$

$N = S + E + I + R$

$$\Longrightarrow \quad \dot{N} = (\gamma - \nu) N - \mu I$$

no equilibria

Model for Proportions

$$N = S + E + I + R$$

"occupancy measure"

$$\longrightarrow \boxed{x = \frac{S}{N}} \longrightarrow \boxed{y = \frac{E}{N}} \longrightarrow \boxed{z = \frac{I}{N}} \longrightarrow \boxed{w = \frac{R}{N}}$$

$$\begin{aligned}
\dot{x} &= (\mu - \beta)\, xz - (\gamma + \kappa\omega)\, x + \gamma \\
\dot{y} &= (\beta x + \mu y)\, z - (\gamma + \delta)\, y \\
\dot{z} &= \mu z^2 + \delta y - (\gamma + \mu + \alpha\tau)\, z \\
\dot{w} &= \mu w z - \gamma w + \kappa\omega x + \alpha\tau z
\end{aligned}$$

Sun and Hsieh,
Appl Math Mod, 2010

- **disease free ratio at**

$$(x_0, y_0, z_0, w_0) = \left(\frac{\gamma}{\gamma + \kappa\omega}, 0, 0, \frac{\kappa\omega}{\gamma + \kappa\omega} \right)$$

- **disease free ratio at**

$$(x_0, y_0, z_0, w_0) = \left(\frac{\gamma}{\gamma + \kappa\omega}, 0, 0, \frac{\kappa\omega}{\gamma + \kappa\omega} \right)$$

- **basic reproduction number**

$$R_0 = \frac{\gamma}{\gamma+\kappa\omega} \cdot \frac{\delta}{\gamma+\delta} \cdot \frac{\beta}{\gamma+\mu+\alpha\tau}$$

identical with formula for SEIR model with constant population

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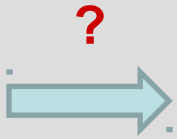


- **if** $R_0 < 1$ **then the DFR is globally asymptotically stable**
- **if** $R_0 > 1$ **there exists a unique, globally stable endemic equilibrium point**

- disease free ratio at

$$(x_0, y_0, z_0, w_0) = \left(\frac{\gamma}{\gamma + \kappa\omega}, 0, 0, \frac{\kappa\omega}{\gamma + \kappa\omega} \right)$$

- basic reproduction number

$$R_0 = \boxed{\frac{\gamma}{\gamma+\kappa\omega}} \cdot \frac{\delta}{\gamma+\delta} \cdot \frac{\beta}{\gamma+\mu+\alpha\tau}$$

identical with formula for SEIR model with constant population

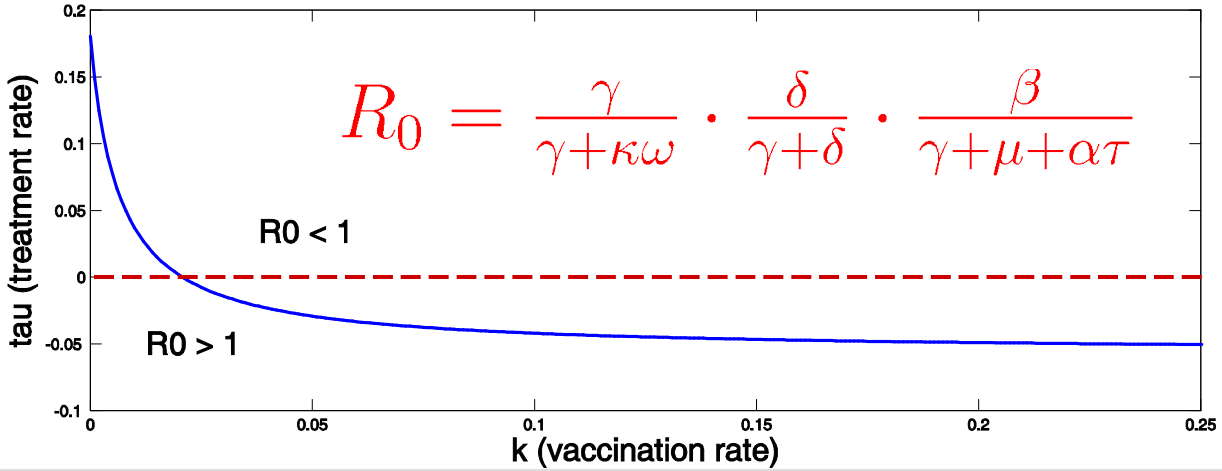
?

- if $R_0 < 1$ then the DFR is globally asymptotically stable
- if $R_0 > 1$ there exists a unique, globally stable endemic equilibrium point

partial results in Sun and Hsieh, Appl Math Mod, 2010

Parameters related to the recent Ebola outbreak in Westafrica (R. Agarwal, M. Balchan and M. Lambros)

parameter	estimated value	interpretation	Reference
γ	0.00683	birth rate of the population	[13]
ν	0.00188	natural death rate of the population	[13]
μ	0.059	disease induced death rate in the population	[11, 12]
β	0.2426	rate of infectiousness of the disease	[11, 12]
δ	0.50	rate at which exposed individuals become infectious	[11, 12]



$R_0 = 3.67$

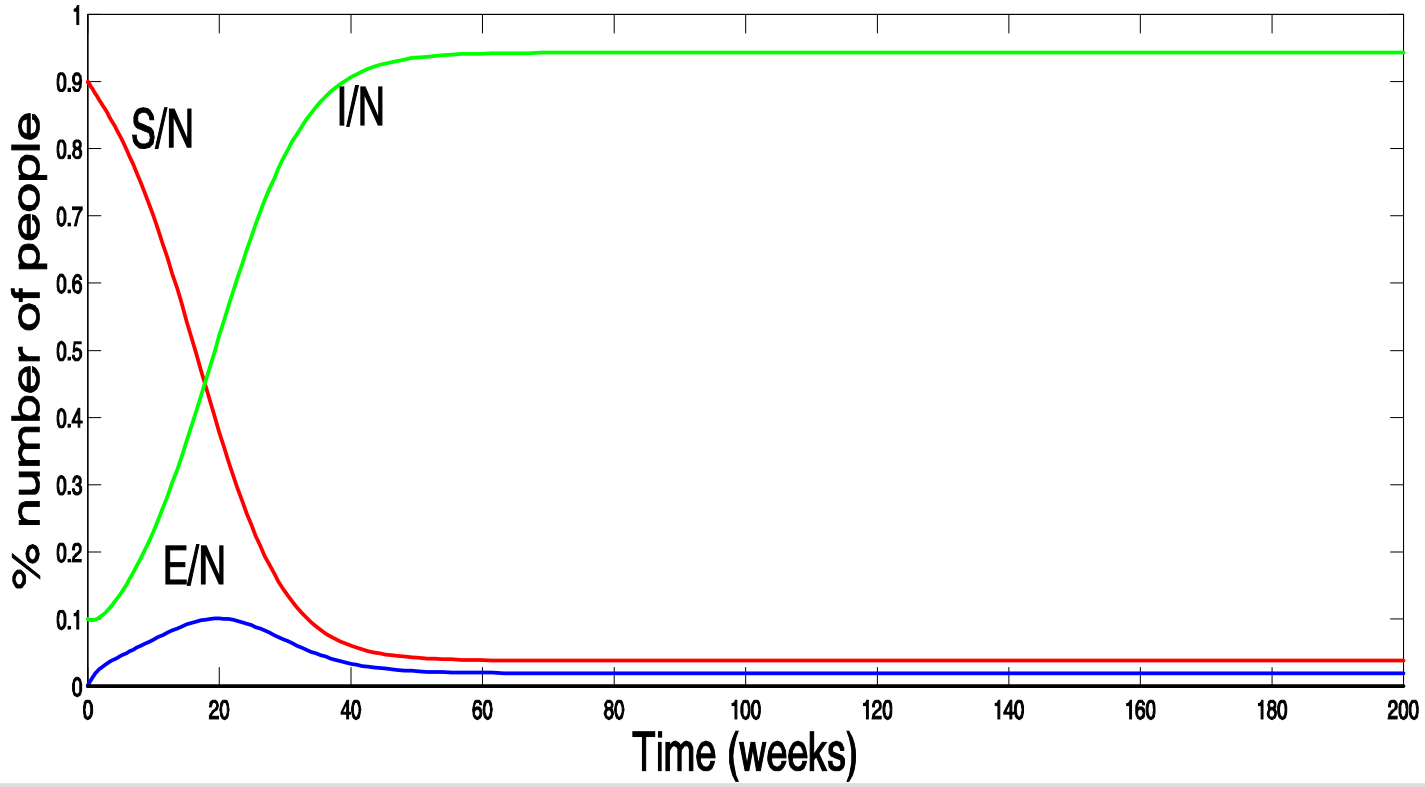
$R_0 = 2.49$

NIH-pub

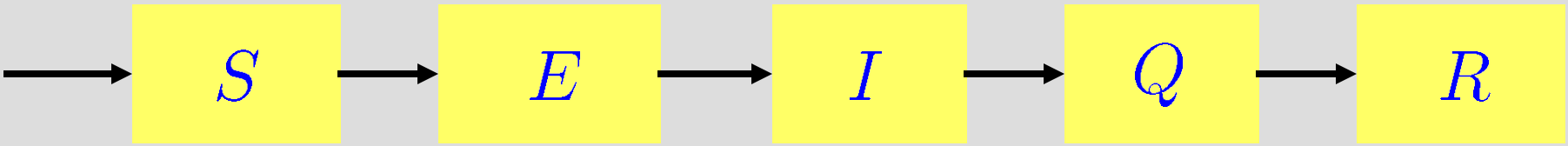
treatment/vaccination would not be effective to control outbreak

Evolution of proportions from

$$(x_0, y_0, z_0, w_0) = (0.90, 0, 0.10, 0)$$



An SEIQR Model with Quarantine



$$
\begin{aligned}
\dot{S}(t) &= \gamma N(t) - \beta \frac{I(t)}{N(t) - Q(t)} S(t) - \nu S(t) - \kappa\omega S(t) \\
\dot{E}(t) &= \beta \frac{I(t)}{N(t) - Q(t)} S(t) - (\delta + \nu)\, E(t) \\
\dot{I}(t) &= \delta E(t) - (\mu + \nu)\, I(t) - \rho I(t) \\
\dot{Q}(t) &= \rho I(t) - (\mu + \nu)\, Q(t) - \alpha\tau Q(t) \\
\dot{R}(t) &= \alpha\tau Q(t) + \kappa\omega S(t) - \nu R(t)
\end{aligned}
$$

$$N = S + E + I + Q + R$$

$$\Longrightarrow \quad \dot{N} = (\gamma - \nu) N - \mu (I + Q)$$

Model for Proportions

$$N = S + E + I + Q + R$$

$$\rightarrow x = \frac{S}{N} \rightarrow y = \frac{E}{N} \rightarrow z = \frac{I}{N} \rightarrow v = \frac{Q}{N} \rightarrow w = \frac{R}{N}$$

$$\begin{aligned}
\dot{x} &= -\beta \frac{xz}{1-v} + \gamma - (\gamma + \kappa\omega)\, x + \mu(z+v)x \\
\dot{y} &= \beta \frac{xz}{1-v} - (\gamma + \delta)\, y + \mu(z+v)y \\
\dot{z} &= \delta y - (\gamma + \mu + \rho)\, z + \mu(z+v)z \\
\dot{v} &= \rho z - (\gamma + \mu + \alpha\tau)\, v + \mu(z+v)v \\
\dot{w} &= \kappa\omega x + \alpha\tau v - \gamma w + \mu(z+v)w
\end{aligned}$$

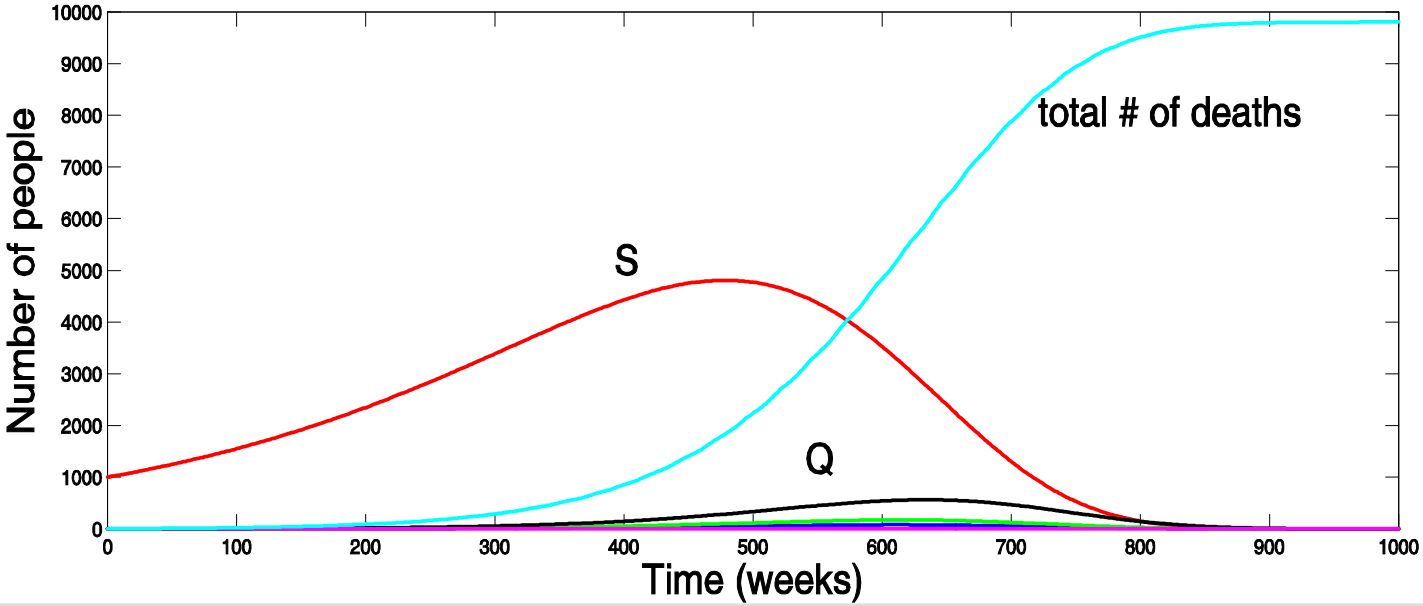
disease free ratio at $(x_0, y_0, z_0, v_0, w_0) = \left(\frac{\gamma}{\gamma + \kappa\omega}, 0, 0, 0, \frac{\kappa\omega}{\gamma + \kappa\omega} \right)$

basic reproduction number

$$R_0 = \frac{\gamma}{\gamma+\kappa\omega} \cdot \frac{\delta}{\gamma+\delta} \cdot \frac{\beta}{\gamma+\mu+\rho}$$

Evolution of the dynamics from

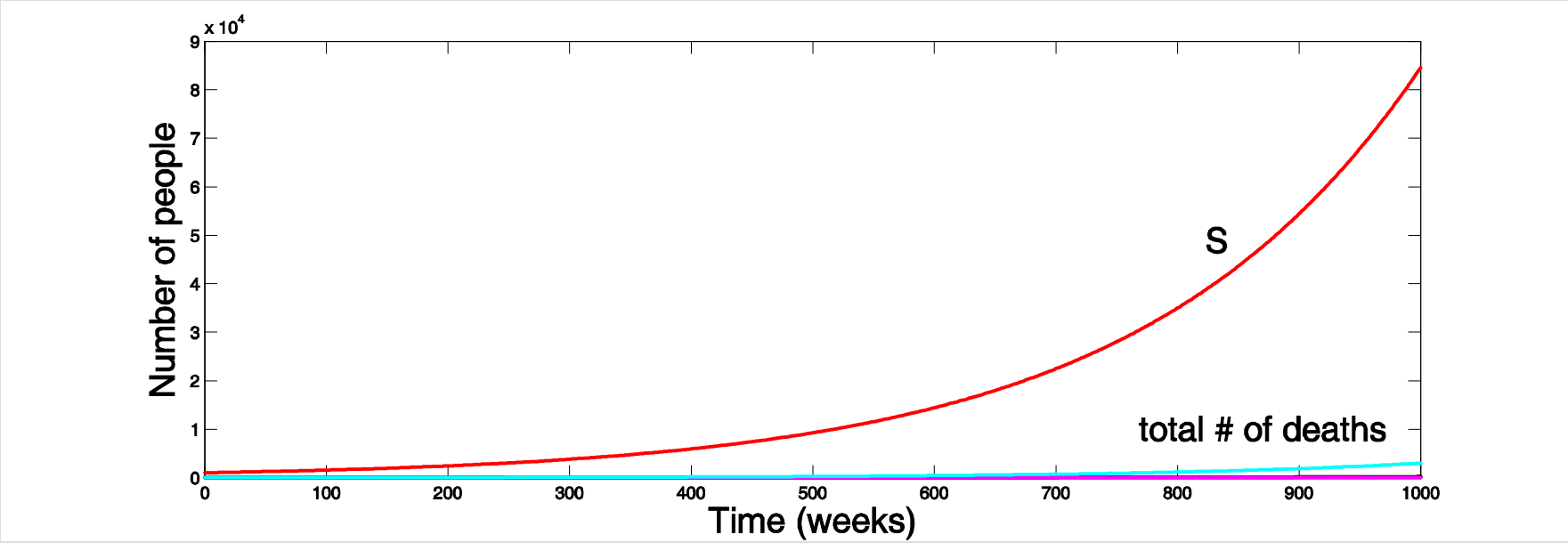
$$(S_0, E_0, I_0, Q_0, R_0) = (999, 0, 1, 0, 0)$$



without vaccination and treatment and quarantine rate $\rho = 0.17$

Evolution of the dynamics from

$$(S_0, E_0, I_0, Q_0, R_0) = (999, 0, 1, 0, 0)$$



without vaccination and treatment and quarantine rate $\rho = 0.182$

Spatial Effects

- **in the model above, individuals are quarantined as they are recognized to have the disease**
- **individuals who were in contact and possibly exposed are not**
- **try to include contact/spatial aspects**

Let $\Omega \subset \mathbb{R}^k$ **model different "locations" and set** $P = N - Q$

$$\begin{aligned}\dot{S}(t,x) &= \gamma N(t,x) - \left(\int_\Omega \beta(x-y)\frac{I(t,y)}{P(t,y)}d\mu(y)\right) S(t,x) \\ &\quad - \left(\int_\Omega \varphi(x-y)\frac{I(t,y)}{P(t,y)}d\mu(y)\right) S(t,x) \\ &\quad + \xi Q_p(t,x) - \nu S(t,x) - \kappa S(t,x)\end{aligned}$$

SIR Model with Constant Population

Let $\Omega \subset \mathbb{R}^k$ model different "locations"/compartments

$$\dot{S}(t,x) = A(x) - \left(\int_\Omega \beta(x-y)I(t,y)d\mu(y)\right) S(t,x)$$

$$+\eta R(t,x) - \nu S(t,x)$$

$$\dot{I}(t,x) = \left(\int_\Omega \beta(x-y)I(t,y)d\mu(y)\right) S(t,x) - (\tau+\nu)S(t,x)$$

$$\dot{R}(t,x) = \tau I(t,x) - (\eta+\nu)R(t,x)$$

$$\Longrightarrow \quad \dot{N}(t,x) = A(x) - \nu N(t,x)$$

SIR model

Let $\Omega \subset \mathbb{R}^k$ model different "locations"/compartments

$$N(t,x) = \frac{A(x)}{\nu} + e^{-\nu t}\left(N(0,x) - \frac{A(x)}{\nu}\right) \to \frac{A(x)}{\nu}$$

total population:

$$\widehat{N}(t) = \int_\Omega N(t,x)d\mu(x) \to \frac{1}{\nu}\int_\Omega A(x)d\mu(x) = \frac{A^*}{\nu}$$

SIR Model with Constant Population

Let $\Omega \subset \mathbb{R}^k$ model different "locations"/compartments

$$
\begin{aligned}
\dot{S}(t,x) &= A(x) - (\beta * I)(t,x)S(t,x) + \eta R(t,x) - \nu S(t,x) \\
\dot{I}(t,x) &= (\beta * I)(t,x)S(t,x) - (\tau + \nu)S(t,x) \\
\dot{R}(t,x) &= \tau I(t,x) - (\eta + \nu)R(t,x)
\end{aligned}
$$

Aggregated populations:

$$\widehat{S}(t) = \int_\Omega S(t,x)d\mu(x)$$

$$\widehat{I}(t) = \int_\Omega I(t,x)d\mu(x) \qquad \widehat{R}(t) = \int_\Omega R(t,x)d\mu(x)$$

$$
\Longrightarrow \quad \dot{\begin{cases}
\widehat{S} &= A^* - \int_\Omega (\beta * I)(t,x)S(t,x)d\mu(x) + \eta\widehat{R} - \nu\widehat{S} \\
\widehat{I} &= \int_\Omega (\beta * I)(t,x)S(t,x)d\mu(x) - (\tau + \nu)\widehat{S} \\
\widehat{R} &= \tau\widehat{I} - (\eta + \nu)\widehat{R}
\end{cases}}
$$

to be continued …

Conclusion

- epidemiological models with time varying populations and significant disease induced death rates are not just a straightforward mathematical extensions of the typical SIR-type models

  there already exists some substantial work (e.g., Sun and Hsieh), but has gaps

- quarantine probably is the most effective means to control a highly infectious disease

  ⇒ models should include these aspects

  ⇒ attempt to model spatial aspects of spread