On a two species chemotaxis system

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Contents

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1 Introduction

Chemotaxis is the ability of microorganisms to respond to chemical signals by moving along the gradient of the chemical substance, either toward the higher concentration (positive taxis) or away from it (negative taxis).

Dictyostelium discoideum (Bacteria)

One species Chemotaxis systems

– Keller and Segel [1970], [1971]

- Jäger and Luckhaus [1992], Biler [1995a] [1995b], [1997]
Herrero-Velázquez [1996], [1997], Velázquez [2002] [2004]
T. Nagai, [1995], T. Nagai, T. Senba and K. Yoshida [1997]
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Dictyostelium discoideum (soil-living amoeba)

Inmune system (white blood cells)

- Macrophages cells
- Neutrophil (granulocyte)

Morphogenesis: the process of formation of the embryo.
See Merkin-Needham-Sleeman [2005], Bollenbach et al [2007],

Angiogenesis

- **Growth of tumours**. See Anderson and Chaplain [1998]
- Formation of embryos
- Healing of skin wound.

Astrophysics and gravitational interaction of particles.
See Biler [1995], Biler-Hilhorst-Nadzieja [1994]

2 Two species chemotaxis systems

We consider a two species system

$$u_t = \underbrace{d_1 \Delta u}_{\text{diffusion}} - \underbrace{\chi_1 \nabla \cdot (u \nabla w)}_{\text{chemotaxis}} + \underbrace{\mu_1 u(1 - u - a_1 v)}_{\text{proliferation and competition}}$$

$$v_t = \underbrace{d_2 \Delta v}_{\text{diffusion}} - \underbrace{\chi_2 \nabla \cdot (v \nabla w)}_{\text{chemotaxis}} + \underbrace{\mu_2 v(1 - a_2 u - v)}_{\text{proliferation and competition}}$$

$$\epsilon w_t = \underbrace{d_w \Delta w}_{\text{diffusion}} - \underbrace{\lambda w}_{\text{degradation}} + \underbrace{g(u, v)}_{\text{production}}$$

Neumann boundary conditions and appropriate initial data in Ω.

$$\epsilon = 0, \quad g(u, v) = u + v, \qquad 0 \leq a_i < 1 \quad \text{for} \;\; i = 1, 2$$

We assume:

$$2(\chi_1 + \chi_2) + a_1 \mu_2 < \mu_1 \quad \text{and} \quad 2(\chi_1 + \chi_2) + a_2 \mu_1 < \mu_2.$$

The limit case

- $\chi_1 = \chi_2 = 0$ is a competitive system of two equations well studied in the literature. See Pao [1981], Cosner and Lazer [1984]

$$u_t = d_1\Delta u + u(e_1 - b_1 u - c_1 v)$$
$$v_t = d_2\Delta v + v(e_2 - b_2 u - c_2 v)$$

  + Neumann boundary conditions. Under assumption

$$\frac{e_1}{c_1} > \frac{e_2}{c_2}, \qquad \frac{e_2}{b_2} > \frac{e_1}{b_1}$$

$$u \longrightarrow \frac{e_1c_2 - c_1e_2}{b_1c_2 - c_1b_2}, \qquad v \longrightarrow \frac{b_1e_2 - e_1b_2}{b_1c_2 - c_1b_2}.$$

  In our case, the assumption is equivalent to $0 < a_i < 1$ (i=1,2).

- $\mu_1 = \mu_2 = 0$. Espejo-Arenas, Stevens and Velázquez [2009]-[2010] Simultaneous and non-simultaneous blow up of both species depending on the parameters and initial mass. See also Espejo-Arenas and Conca [2012], Biler-Espejo-Guerra [2013].

Steady states

$$\begin{cases} 0 = \Delta u - \chi_1 \nabla \cdot (u \nabla w) + \mu_1 u(1 - u - a_1 v) & \text{in } \Omega, \\ 0 = \Delta v - \chi_2 \nabla \cdot (v \nabla w) + \mu_2 v(1 - a_2 u - v) & \text{in } \Omega, \\ 0 = \Delta w - \lambda w + u + v & \text{in } \Omega, \\ \frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = \frac{\partial v}{\partial n} = 0, & \text{in } \partial\Omega \end{cases}$$

The unique positive and bounded steady states is given by

$$u^\star \equiv \frac{1 - a_1}{1 - a_1 a_2}, \qquad v^\star \equiv \frac{1 - a_2}{1 - a_1 a_2}.$$

An Auxiliary System of ODEs

$$
\begin{cases}
\overline{u}' = \overline{u}\,[\mu_1 - (\mu_1 - \chi_1)\overline{u} - \chi_1\underline{u} + \chi_1\overline{v} - (\chi_1 + \mu_1 a_1)\underline{v}], & t > 0, \\
\underline{u}' = \underline{u}\,[\mu_1 - \chi_1\overline{u} - (\mu_1 - \chi_1)\underline{u} - (\chi_1 + \mu_1 a_1)\overline{v} + \chi_1\underline{v}], & t > 0, \\
\overline{v}' = \overline{v}\,[\mu_2 + \chi_2\overline{u} - (\chi_2 + \mu_2 a_2)\underline{u} - (\mu_2 - \chi_2)\overline{v} - \chi_2\underline{v}], & t > 0, \\
\underline{v}' = \underline{v}\,[\mu_2 - (\chi_2 + \mu_2 a_2)\overline{u} + \chi_2\underline{u} - \chi_2\overline{v} - (\mu_2 - \chi_2)\underline{v}], & t > 0,
\end{cases}
$$

with initial conditions

$$\overline{u}(0) = \overline{u}_0, \quad \underline{u}(0) = \underline{u}_0, \quad \overline{v}(0) = \overline{v}_0, \quad \text{and} \quad \underline{v}(0) = \underline{v}_0.$$

The steady states of the ODEs system are given by

$$\overline{u}^\star = \underline{u}^\star = u^\star \equiv \frac{1 - a_1}{1 - a_1 a_2}, \qquad \overline{v}^\star = \underline{v}^\star = v^\star \equiv \frac{1 - a_2}{1 - a_1 a_2}$$

We analize the system of ODEs. under assumptions

$$0 < \underline{u}_0 < u^* < \overline{u}_0 < \infty, \qquad 0 < \underline{v}_0 < v^* < \overline{v}_0 < \infty,$$

Step 1a.- $0 < \underline{u} < \overline{u} < \infty$ for $t \in (0, \infty)$;

Step 1b.- $0 < \underline{v} < \overline{v} < \infty$ for $t \in (0, \infty)$;

Step 2a.- $0 < \underline{u} < u^* < \overline{u} < \infty$ for $t \in (0, \infty)$;

Step 2b.- $0 < \underline{v} < v^* < \overline{v} < \infty$ for $t \in (0, \infty)$;

Step 3.- $\lim_{t\to\infty} |\underline{u} - \overline{u}| + |\underline{v} - \overline{v}| \longrightarrow 0.$

Idea of Step 3.

$$\frac{d}{dt} \log \frac{\overline{u}}{\underline{u}} = \frac{\overline{u}_t}{\overline{u}} - \frac{\underline{u}_t}{\underline{u}} = -(\mu_1 - 2\chi_1)(\overline{u} - \underline{u}) + (2\chi_1 + \mu_1 a_1)(\overline{v} - \underline{v})$$

and

$$\frac{d}{dt} \log \frac{\overline{v}}{\underline{v}} = (2\chi_2 + \mu_2 a_2)(\overline{u} - \underline{u}) - (\mu_2 - 2\chi_2)(\overline{v} - \underline{v}) \qquad \text{for all } t > 0.$$

We add both to obtain

$$\frac{d}{dt}\left(\log\frac{\overline{u}}{\underline{u}}+\log\frac{\overline{v}}{\underline{v}}\right)=(-\mu_1+2(\chi_1+\chi_2)+\mu_2a_2)(\overline{u}-\underline{u})+$$

$$(-\mu_2+2(\chi_1+\chi_2)+\mu_1a_1)(\overline{v}-\underline{v})$$

for

$$\varepsilon:=\min\left\{\mu_1-2(\chi_1+\chi_2)-\mu_2a_2\,,\ \mu_2-2(\chi_1+\chi_2)-\mu_1a_1\right\}$$

then

$$\frac{d}{dt}\left(\log\frac{\overline{u}}{\underline{u}}+\log\frac{\overline{v}}{\underline{v}}\right)\leq-\varepsilon(\overline{u}-\underline{u})-\varepsilon(\overline{v}-\underline{v})\qquad\text{for all } t>0,$$

This first entails that

$$\log\frac{\overline{u}}{\underline{u}}\leq\log\frac{\overline{u}_0}{\underline{u}_0}+\log\frac{\overline{v}_0}{\underline{v}_0}:=c_0\qquad\text{for all } t>0,$$

and

$$\log\frac{\overline{u}^*}{\underline{u}}\leq c_0\Longrightarrow\underline{u}\geq\overline{u}^*e^{-c_0}\geq\overline{u}^*\,\frac{\underline{u}_0}{\overline{u}_0}\,\frac{\underline{v}_0}{\overline{v}_0}>0\qquad\text{for all } t>0.$$

In the same way

$$\underline{v} \geq \overline{v}^* \, \frac{\underline{u}_0}{\overline{u}_0} \, \frac{\underline{v}_0}{\overline{v}_0} > 0 \qquad \text{for all } t > 0.$$

By the Mean Value Theorem

$$\overline{u}(t) - \underline{u}(t) = e^{\xi_1(t)} \left(\log \overline{u}(t) - \log \underline{u}(t) \right) \qquad \overline{v}(t) - \underline{v}(t) = e^{\xi_2(t)} \left(\log \overline{v}(t) - \log \underline{v}(t) \right).$$

and

$$\frac{d}{dt}\left(\log \frac{\overline{u}}{\underline{u}} + \log \frac{\overline{v}}{\underline{v}} \right) \leq -\varepsilon_0 \left(\log \frac{\overline{u}}{\underline{u}} + \log \frac{\overline{v}}{\underline{v}} \right) \qquad \text{for all } t > 0$$

is valid with

$$\varepsilon_0 = \varepsilon \, \frac{\underline{u}_0}{\overline{u}_0} \, \frac{\underline{v}_0}{\overline{v}_0} \, \min\{\overline{u}^*, \overline{v}^*\}.$$

After routine computations

$$0 < \log \frac{\overline{u}}{\underline{u}} \leq e^{-\varepsilon_0 t} \log \frac{\overline{u}_0 \overline{v}_0}{\underline{u}_0 \underline{v}_0}, \qquad 0 < \log \frac{\overline{v}}{\underline{v}} \leq e^{-\varepsilon_0 t} \log \frac{\overline{u}_0 \overline{v}_0}{\underline{u}_0 \underline{v}_0}, \qquad \text{for all } t > 0$$

and thereby shows that

$$|\overline{u}(t) - \underline{u}(t)| + |\overline{v}(t) - \underline{v}(t)| \to 0 \qquad \text{as } t \to \infty.$$

which ends the analysis of the ODE system.

Comparison: PDE system- ODE system

Reaction Diffusion Systems: Rectangle Method see Pao [1981].

We prove that, under assumption

$$0 < \underline{u}_0 \leq u_0(x) \leq \overline{u}_0 \quad \text{and} \quad 0 < \underline{v}_0 \leq v_0(x) \leq \overline{v}_0 \qquad \text{for all } x \in \Omega.$$

we have

$$\underline{u} \leq u \leq \overline{u} \quad \text{and} \quad \underline{v} \leq v \leq \overline{v}.$$

We construct the functions:

$$\overline{U}(x,t) := u(x,t) - \overline{u}(t), \qquad \underline{U}(x,t) := u(x,t) - \underline{u}(t),$$

$$\overline{V}(x,t) := v(x,t) - \overline{v}(t), \qquad \underline{V}(x,t) := v(x,t) - \underline{v}(t)$$

which satisfy

$$\overline{U}_t - \Delta \overline{U} + \chi_1 \nabla \overline{U} \nabla w \; = \; \overline{U} \; [\mu_1 + (\chi_1 - \mu_1)(u + \overline{u}) + (\chi_1 - \mu_1 a_1) v - \chi_1 \lambda w] \\ + \chi_1 \overline{u}\,\overline{V} - \mu_1 a_1 \overline{u}\,\underline{V} + \chi_1 (\underline{u} + \underline{v} - \lambda w).$$

we multiply by $\overline{U}_+$ and after integration

$$\frac{d}{dt}\frac{1}{2}\int_\Omega \overline{U}_+^2 + \int_\Omega |\nabla \overline{U}_+|^2 = -\frac{\chi_1}{2}\int_\Omega \nabla \overline{U}_+^2 \nabla w + \int_\Omega b(x,t)\overline{U}_+^2$$

$$+\chi_1 \int_\Omega \overline{u}\,\overline{V}\,\overline{U}_+ - \mu_1 a_1 \int_\Omega \overline{u}\,\underline{V}\,\overline{U}_+ + \chi_1 \int_\Omega (\underline{u} + \underline{v} - \lambda w)\overline{U}_+$$

we obtain similar expressions for $\underline{U}$, $\overline{V}$, $\underline{V}$.

After many routine computations we obtain, for any $t < T$

$$\frac{d}{dt}\int_\Omega \left(\overline{U}_+^2 + \underline{U}_-^2 + \overline{V}_+^2 + \underline{V}_+^2\right) \leq k(T)\int_\Omega \left(\overline{U}_+^2 + \underline{U}_-^2 + \overline{V}_+^2 + \underline{V}_-^2\right)$$

Gronwall′s lemma gives

$$\overline{U}_+ = \underline{U}_- = \overline{V}_+ = \underline{V}_- = 0.$$

3.- Competitive Exclusion (Stinner-T-winkler 2013)

We consider the system

$$u_t = \underbrace{d_1 \Delta u}_{\text{diffusion}} - \underbrace{\chi_1 \nabla \cdot (u \nabla w)}_{\text{chemotaxis}} + \underbrace{\mu_1 u(1 - u - a_1 v)}_{\text{proliferation and competition}}$$

$$v_t = \underbrace{d_2 \Delta v}_{\text{diffusion}} - \underbrace{\chi_2 \nabla \cdot (v \nabla w)}_{\text{chemotaxis}} + \underbrace{\mu_2 v(1 - a_2 u - v)}_{\text{proliferation and competition}}$$

$$0 = \underbrace{d_w \Delta w}_{\text{diffusion}} - \underbrace{\lambda w}_{\text{degradation}} + \underbrace{ku + v}_{\text{production}}$$

Neumann boundary conditions and appropriate initial data in Ω.

We define the new parameters

$$q_1 := \frac{\chi_1}{\mu_1} \qquad \text{and} \qquad q_2 := \frac{\chi_2}{\mu_2}.$$

We consider the assumptions

$$a_1 > 1 > a_2$$

k, q_1 and q_2 are nonnegative and $q_1 = \frac{\chi_1}{\mu_1} \leq a_1, q_2 = \frac{\chi_2}{\mu_2} < \frac{1}{2}$ and

$$kq_1 + \max\{q_2\ ,\ \frac{a_2 - a_2 q_2}{1 - 2q_2}\ ,\ \frac{kq_2 - a_2 q_2}{1 - 2q_2}\} < 1.$$

The assumptions are equivalent to $kq_1 + q_2 < 1$ and

$$\begin{cases} kq_1 + (2 - a_2)q_2 + a_2 - 2kq_1q_2 < 1 & \text{if } kq_2 < a_2, \\ kq_1 + (2 - a_2 + k)q_2 - 2kq_1q_2 < 1 & \text{if } kq_2 \geq a_2. \end{cases}$$

Then, we have that

$$u \longrightarrow 0, \qquad v \longrightarrow 1$$

Notice that the assumptions in the prototypical case $\chi_1 = \chi_2$, $\mu_1 = \mu_2$ are reduced to

$$\frac{\chi}{\mu} < \begin{cases} \frac{2+k-a_2-\sqrt{(k+2-a_2)^2-8k(1-a_2)}}{4k} & \text{if } a_2 > kq \\ \frac{2+2k-a_2-\sqrt{(2k+2-a_2)^2-8k}}{4k} & \text{if } a_2 \leq kq. \end{cases}$$

If we moreover have $k = 1$ then

$$\frac{\chi}{\mu} < \begin{cases} \frac{4-a_2-\sqrt{8-8a_2+a_2^2}}{4} & \text{if } a_2 \leq q \\ \frac{1-a_2}{2} & \text{if } a_2 > q. \end{cases}$$

In the limit case $k = 0$,

$$\frac{\chi}{\mu} < \frac{1-a_2}{2-a_2}$$

The borderline case $a_2 = 0$ reads

$$\frac{\chi}{\mu} < \frac{1}{2}$$

already found in T-winkler [2007]

Non-local terms (Negreanu-T [2013])

$$\begin{cases} u_t - \Delta u = -\chi_1 \nabla \cdot (u \nabla w) + u \left(a_0 - a_1 u - a_2 v - a_3 \int_\Omega u - a_4 \int_\Omega v \right), \\ v_t - \Delta v = -\chi_2 \nabla \cdot (v \nabla w) + v \left(b_0 - b_1 u - b_2 v - b_3 \int_\Omega u - b_4 \int_\Omega v \right), \\ -\Delta w + \lambda w = f + k_1 u + k_2 v, \end{cases}$$

with the homogeneous Neumann boundary conditions

$$\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0, \qquad x \in \partial\Omega,\ t > 0,$$

and initial data

$$u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \qquad x \in \Omega.$$

“Global Competition”

Members of one species u compete for a limited resource z satisfying

$$u_t - \Delta u + \mu u = zu, \qquad x \in \Omega, \quad t > 0$$

KPP-Fisher equation $z = (1 - u)$.

We assume that the resources diffuse and degrade with large diffusion coefficient ϵ^{-1}

$$-\frac{1}{\epsilon}\Delta z_\epsilon + \alpha_1 z_\epsilon = 1 - \alpha_2 u \qquad x \in \Omega$$

$$\frac{\partial z_\epsilon}{\partial \vec{n}} = 0.$$

After Integration we have

$$\int_\Omega z_\epsilon = \frac{|\Omega|}{\alpha_1} - \frac{\alpha_2}{\alpha_1}\int_\Omega u.$$

Multiplying by u and after integration and thanks to Young inequality we have

$$\int_\Omega |\nabla z_\epsilon|^2 dx \leq \epsilon(1 + c(\alpha_1, \alpha_2) \int_\Omega u^2) \to 0 \quad \text{as } \epsilon \to 0.$$

Then

$$z_\epsilon \longrightarrow constant := \frac{|\Omega|}{\alpha_1} - \frac{\alpha_2}{\alpha_1} \int_\Omega u.$$

the equation

$$u_t - \Delta u + \mu u = zu, \qquad x \in \Omega, \quad t > 0$$

is replace by

$$u_t - \Delta u + \mu u = \mu u(1 - a_3 \int_\Omega u), \qquad x \in \Omega, \quad t > 0.$$

If $\alpha_2 = \alpha_2(x)$ then, the nonlocal term $\int_\Omega \alpha_3(x) u$.

For the two species chemotaxis system we consider

$$-\frac{1}{\epsilon}\Delta z_\epsilon + \alpha_1 z_\epsilon = 1 - \alpha_2 u - \alpha_3 v \qquad x \in \Omega \quad + \quad NBC$$

$$z_\epsilon \to \alpha - a_3 \int_\Omega u - a_4 \int_\Omega v.$$

Under assumptions

$$\int_0^\infty |\sup_{x\in\Omega} f - \inf_{x\in\Omega} f| \le C_0 < \infty.$$

$$\chi_1,\ \chi_2,\ k_1,\ k_2,\ a_i,\ b_i > 0, \quad \text{for} \ \ i = 1, 2,$$

$$a_i \in I\!R,\ b_i \in I\!R, \quad \text{for} \ \ i = 3, 4$$

$$a_1 > 2k_1(\chi_1 + \chi_2) + b_1 + |b_3| + |a_3| \ \text{ and } \ b_2 > 2k_2(\chi_1 + \chi_2) + a_1 + |a_4| + |b_4|$$

we obtain the asymptotic behavior for positive initial data

$$u(\cdot, t) \longrightarrow u^* \equiv \frac{a_0(b_2 + b_4) - b_0(a_2 + a_4)}{(b_2 + b_4)(a_1 + a_3) - (b_1 + b_3)(a_2 + a_4)}$$

$$v(\cdot, t) \longrightarrow v^* \equiv \frac{a_0(b_1 + b_3) - b_0(a_1 + a_3)}{(b_1 + b_3)(a_2 + a_4) - (b_2 + b_4)(a_1 + a_3)}.$$

Two species Parabolic-ODE chemotactic system

$$u_t = \underbrace{d_1 \Delta u}_{\text{diffusion}} - \underbrace{\nabla(u \chi_1(w) \nabla w)}_{\text{chemotaxis}}$$

$$v_t = \underbrace{d_2 \Delta v}_{\text{diffusion}} - \underbrace{\nabla(v \chi_2(w) \nabla w)}_{\text{chemotaxis}}$$

$$w_t = h(u, v, w)$$

We assume

$$\chi_i,\ h \in W^{1,\infty}_{loc}(I\!R^2_+ \times I\!R), \qquad \chi_i > 0.$$

$$\frac{\partial h}{\partial u} \geq \epsilon_u > 0 \quad \text{and} \quad \frac{\partial h}{\partial v} \geq \epsilon_v > 0$$

$$\frac{\partial h}{\partial w} < 0.$$

There exists w^* such that

$$h(u^*, v^*, w^*) = 0$$

where

$$u^* = \frac{1}{|\Omega|} \int_\Omega u_0 dx, \qquad v^* = \frac{1}{|\Omega|} \int_\Omega v_0 dx.$$

Consequently (u^*, v^*, w^*) is a constant stationary solution of the system.

Global existence of solutions.

We assume

$$-h(0,0,w) \le \frac{k_i}{\chi_i(w)} \text{ for some } k_i > 0,$$

$$0 < k_{0i} \le \chi_i(w) e^{\int_{\underline{w}}^{w} \chi_i(s)ds} \qquad \text{for } w > \underline{w},$$

for $k_{0i} > 0$, with $i = 1, 2$. There exists $\overline{u}$ and $\overline{v}$ such that

$$h(u,v,\underline{w}) \ge 0, \qquad h(u,v,\overline{w}) \le 0, \quad \text{for} \quad 0 \le u \le \overline{u}, \quad 0 \le v \le \overline{v},$$

where

$$\overline{u} := f_1(\overline{w}) \max\left\{k_1 \left(\epsilon_u k_{01}\right)^{-1}, \|u_0\|_{L^\infty(\Omega)}\right\},$$

$$\overline{v} := f_2(\overline{w}) \max\left\{k_2 \left(\epsilon_v k_{02}\right)^{-1}, \|v_0\|_{L^\infty(\Omega)}\right\},$$

for f_i defined by

$$f_i(w) = e^{\int_{\underline{w}}^{w} \chi_i(s)ds} \qquad i = 1, 2.$$

Then, by using an iterative method (Alikakos-Mosher iteration) we have global boundedness.

Stability of the homogeneous steady states.

We consider

$$\textit{There exists } \alpha \in (0,1) \textit{ such that}$$

$$\alpha h_w + u h_u \chi_1 + v h_v \chi_2 < 0 \quad \text{and} \quad 2\sqrt{1-\alpha}\, h_w + u h_v \chi_1 + v h_u \chi_2 < 0.$$

Using a Lyapunov functional, we get that the steady state is globally asymptotically stable.

Notice that the previous assumptions are satisfied, for instance for

$$h(u,v,w) = u + v - w, \qquad \chi_i(w) = \frac{\gamma_i}{1+\gamma_i w} \quad \text{for } \gamma_i < 1/4$$

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Thank you very much for your attention!